MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

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DEPARTMENT OF MECHANICAL ENGINEERING

DIGITAL NOTES OF DYNAMICS OF MACHINERY

For

B.TECH - II YEAR – II SEMESTER

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UNIT-I

GYROSCOPIC COUPLE AND PRECESSIONAL MOTION

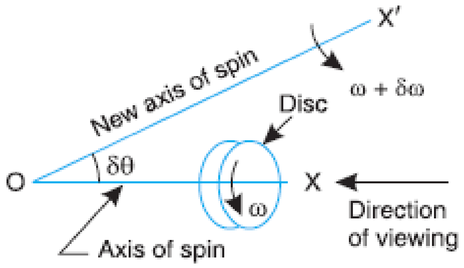
Introduction

1. When a body moves along a curved path with a uniform linear velocity, a force in the direction of centripetal acceleration (known as centripetal force) has to be applied externally over the body, so that it moves along the required curved path. This external force applied is known as **active force.**

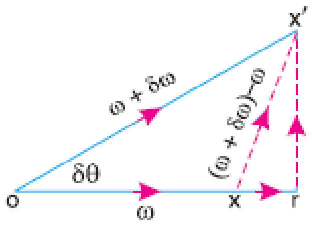
2. When a body, itself, is moving with uniform linear velocity along a circular path, it is subjected to the centrifugal force* radially outwards. This centrifugal force is called **reactive force.** The action of the reactive or centrifugal force is to tilt or move the body along radially outward direction.

Precessional Angular Motion

We have already discussed that the angular acceleration is the rate of change of angular velocity with respect to time. It is a vector quantity and may be represented by drawing a vector diagram with the help of right hand screw rule.



(*a*)



(*b*)

Consider a disc, as shown in Fig (*a*), revolving or spinning about the axis *OX* (known as **axis of spin**) in anticlockwise when seen from the front, with an angular velocity ω in a plane at right angles to the paper.

After a short interval of time δ*t*, let the disc be spinning about the new axis of spin *OX* ′(at an angle δθ) with an angular velocity (ω+ δω). Using the right hand screw rule, initial angular velocity of the disc (ω) is represented by vector *ox*; and the final angular velocity of the disc (ω+ δω) is represented by vector *ox*′ as shown in Fig. 14.1 (*b*). The vector *xx*′ represents the change of angular velocity in time δ*t* *i.e.* the angular acceleration of the disc. This may be resolved into two components,
one parallel to *ox* and the other perpendicular to *ox.*

Component of angular acceleration in the direction of *ox,*

$$\alpha_t = \frac{xr}{\delta t} = \frac{or - ox}{\delta t} = \frac{ox' \cos \delta\theta - ox}{\delta t}$$

$$= \frac{(\omega + \delta\omega)\ \cos \delta\theta - \omega}{\delta t} = \frac{\omega \cos \delta\theta + \delta\omega \cos \delta\theta - \omega}{\delta t}$$

Since δθ is very small, therefore substituting cos δθ = 1, we have

$$\alpha_t = \frac{\omega + \delta\omega - \omega}{\delta t} = \frac{\delta\omega}{\delta t}$$

In the limit, when $\delta t \to 0$,

$$\alpha_t = \underset{\delta t \to 0}{\mathrm{Lt}} \left(\frac{\delta\omega}{\delta t}\right) = \frac{d\omega}{dt}$$

Component of angular acceleration in the direction perpendicular to *ox*,

$$\alpha_c = \frac{rx'}{\delta t} = \frac{ox' \sin\delta\theta}{\delta t} = \frac{(\omega + \delta\omega)\sin\delta\theta}{\delta t} = \frac{\omega\sin\delta\theta + \delta\omega.\sin\delta\theta}{\delta t}$$

Since δθ in very small, therefore substituting sin δθ = δθ, we have

$$\alpha_c = \frac{\omega.\delta\theta + \delta\omega.\delta\theta}{\delta t} = \frac{\omega.\delta\theta}{\delta t}$$

...(Neglecting δω.δθ, being very small)

In the limit when $\delta t \to 0$,

$$\alpha_c = \underset{\delta t \to 0}{\mathrm{Lt}} \frac{\omega.\delta\theta}{\delta t} = \omega \times \frac{d\theta}{dt} = \omega.\omega_P \qquad \ldots\left(\text{Substituting } \frac{d\theta}{dt} = \omega_P\right)$$

∴ Total angular acceleration of the disc

$$= \text{vector } xx' = \text{vector sum of } \alpha_t \text{ and } \alpha_c$$

$$= \frac{d\omega}{dt} + \omega \times \frac{d\theta}{dt} = \frac{d\omega}{dt} + \omega.\omega_P$$

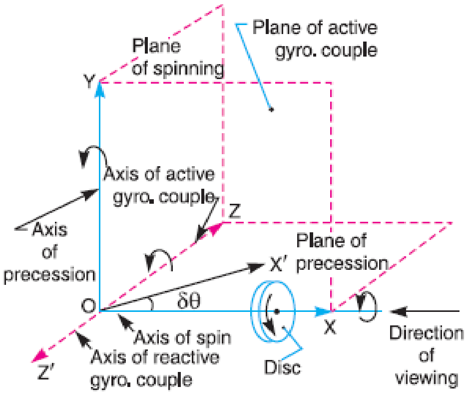
where dθ/dt is the angular velocity of the axis of spin about a certain axis, which is perpendicular to the plane in which the axis of spin is going to rotate. This angular velocity of the axis of spin (i.e.dθ/dt) is known as angular velocity of precession and is denoted by ω_P. The axis, about which the axis of spin is to turn, is known as axis of precession. The angular motion of the axis of spin about the axis of precession is known as precessional angular motion.

Gyroscopic Couple

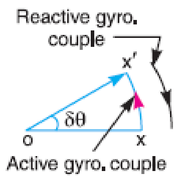
Consider a disc spinning with an angular velocity ω rad/s about the axis of spin OX, in anticlockwise direction when seen from the front, as shown in Fig. 14.2 (a). Since the plane in which the disc is rotating is parallel to the plane YOZ, therefore it is called plane of spinning. The plane XOZ is a horizontal plane and the axis of spin rotates in a plane parallel to the horizontal plane about an axis OY. In other words, the axis of spin is said to be rotating or processing about an axis OY. In other words, the axis of spin is said to be rotating or processing about an axis OY (which is perpendicular to both the axes OX and OZ) at an angular velocity _P rap/s. This horizontal plane XOZ is called plane of precession and OY is the axis of precession.

Let *I* = Mass moment of inertia of the disc about *OX*, and
ω= Angular velocity of the disc.
Angular momentum of the disc = *I*. ω

Since the angular momentum is a vector quantity, therefore it may be represented by the vector *ox*′ , as shown in Fig. 14.2 (*b*). The axis of spin *OX* is also rotating anticlockwise when seen from the top about the axis *OY*. Let the axis *OX* is turned in the plane *XOZ* through a small angle δθ radians to the position *OX*′ , in time δ*t* seconds. Assuming the angular velocity ω to be constant, the angular momentum will now be represented by vector *ox*′.



(a)



(b)

Change in angular momentum

$$= \vec{ox'} - \vec{ox} = \vec{xx'} = \vec{ox}.\delta\theta \qquad \text{...(in the direction of } \vec{xx'}\text{)}$$
$$= I.\,\omega.\delta\theta$$

and rate of change of angular momentum

$$= I.\omega \times \frac{\delta\theta}{dt}$$

Since the rate of change of angular momentum will result by the application of a couple to the disc, therefore the couple applied to the disc causing precession,

$$C = \underset{\delta t \to 0}{\mathrm{Lt}}\ I.\omega \times \frac{\delta\theta}{\delta t} = I.\omega \times \frac{d\theta}{dt} = I.\omega.\omega_P \qquad \ldots\left(\because \frac{d\theta}{dt} = \omega_P\right)$$

where ω_P = Angular velocity of precession of the axis of spin or the speed of rotation of the axis of spin about the axis of precession *OY*.

1. The couple $I.\omega.\omega_p$, in the direction of the vector *xx′* (representing the change in angular momentum) is the *active gyroscopic couple*, which has to be applied over the disc when the axis of spin is made to rotate with angular velocity ω_P about the axis of precession. The vector *xx′* lies in the plane *XOZ* or the horizontal plane. In case of a very small displacement δθ, the vector *xx′* will be perpendicular to the vertical plane *XOY*. Therefore the couple causing this change in the angular momentum will lie in the plane *XOY*. The vector *xx′*, as shown in Fig(*b*), represents an anticlockwise couple in the plane *XOY*. Therefore, the plane *XOY* is called the *plane of active gyroscopic couple* and the axis *OZ* perpendicular to the plane *XOY*, about which the couple acts, is called the axis of active gyroscopic couple.

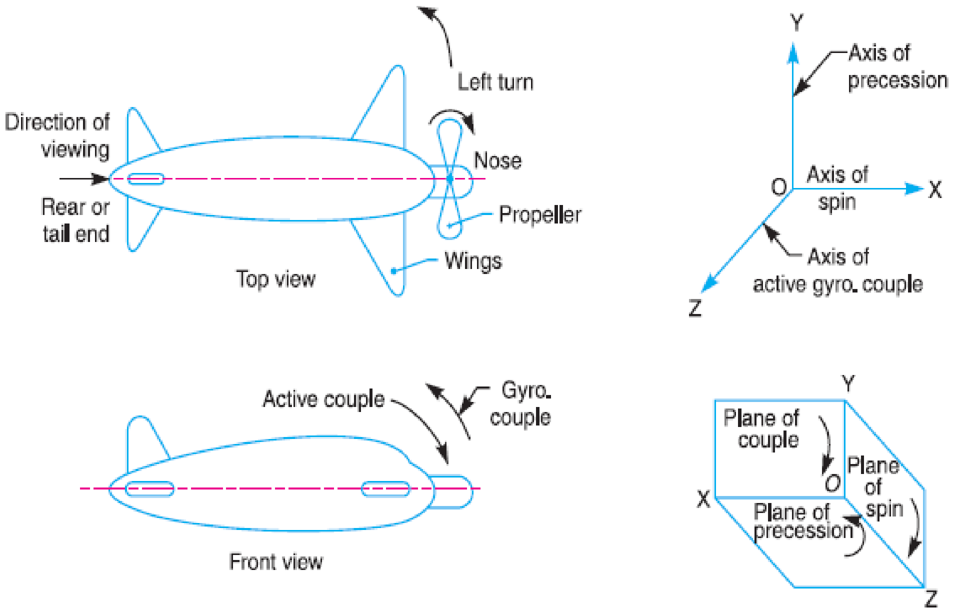
2. When the axis of spin itself moves with angular velocity ω_P, the disc is subjected to *reactive couple* whose magnitude is same (*i.e.* $I.\omega.\omega_P$) but opposite in direction to that of active couple. This reactive couple to which the disc is subjected when the axis of spin rotates about the axis of precession is known as *reactive gyroscopic couple*. The axis of the reactive gyroscopic couple is represented by *OZ′* in Fig(*a*).

3. The gyroscopic couple is usually applied through the bearings which support the shaft. The bearings will resist equal and opposite couple.
4. The gyroscopic principle is used in an instrument or toy known as gyroscope. The gyroscopes are installed in ships in order to minimize the rolling and pitching effects of waves. They are also used in Aeroplanes, monorail cars, gyrocompasses etc.

Effect of the Gyroscopic Couple on an Aeroplane

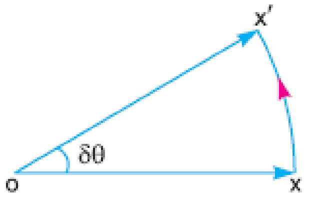
The top and front view of an aeroplane are shown in Fig (*a*). Let engine or propeller rotates in the clockwise direction when seen from the rear or tail end and the aeroplane takes a turn to the left.

Let ω= Angular velocity of the engine in rad/s,
m = Mass of the engine and the propeller in kg,
k = Its radius of gyration in metre
I = Mass moment of inertia of the engine and the propeller in kg-m^2 = $m.k^2$,
v = Linear velocity of the aeroplane in m/s,
R = Radius of curvature in metres, and
ω_P = Angular velocity of precession=V/R
Gyroscopic couple acting on the aeroplane, $C = I.\omega.\omega_P$

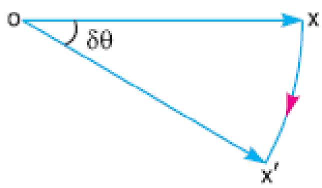


Before taking the left turn, the angular momentum vector is represented by *ox*. When it takes left turn, the active gyroscopic couple will change the direction of the angular momentum vector from *ox* to *ox*′ as shown in Fig(*a*). The vector *xx*′,in the limit, represents the change of angular momentum or the active gyroscopic couple and is perpendicular to *ox*. Thus the plane of active gyroscopic couple *XOY* will be perpendicular to *xx*′, *i.e.* vertical in this case, as shown in Fig (*b*). By applying right hand screw rule to vector *xx*′, we find that the direction of active gyroscopic couple is clockwise as shown in the front view of Fig (*a*).

In other words, for left hand turning, the active gyroscopic couple on the aeroplane in the axis *OZ* will be clockwise as shown in Fig (*b*).The reactive gyroscopic couple (equal in magnitude of active gyroscopic couple) will act in the opposite direction (*i.e.* in the anticlockwise direction) and the effect of this couple is, therefore, to raise the nose and dip the tail of the aeroplane.



(*a*) Aeroplane taking left turn.

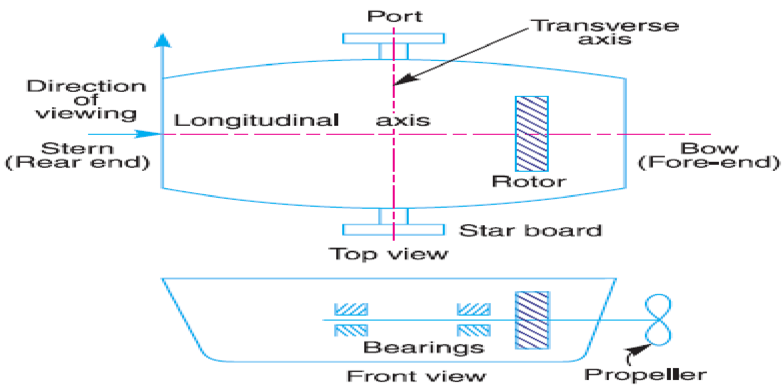


(*b*) Aeroplane taking right turn.

Terms Used in a Naval Ship

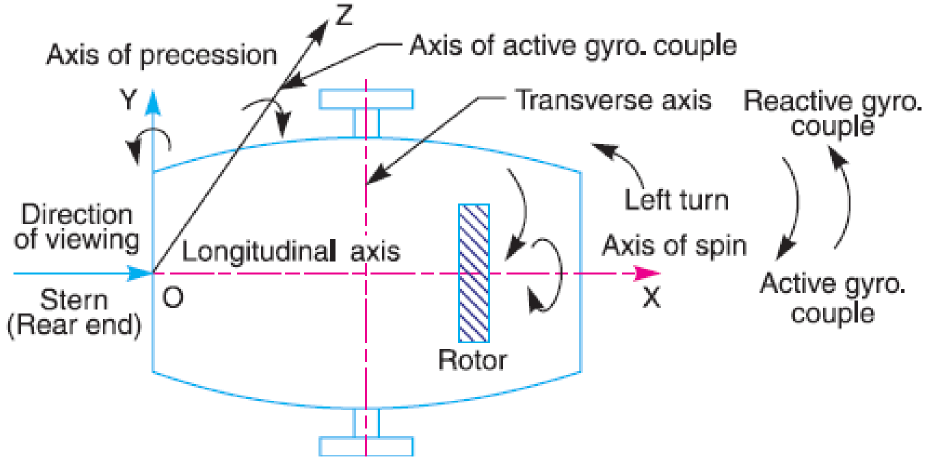
The top and front views of a naval ship are shown in Fig 14.7. The fore end of the ship is called ***bow*** and the rear end is known as ***stern*** or ***aft***. The left hand and right hand sides of the ship, when viewed from the stern are called ***port*** and ***star-board*** respectively. We shall now discuss the effect of gyroscopic couple on the naval ship in the following three cases:

1. Steering,
2. Pitching,
3. Rolling.

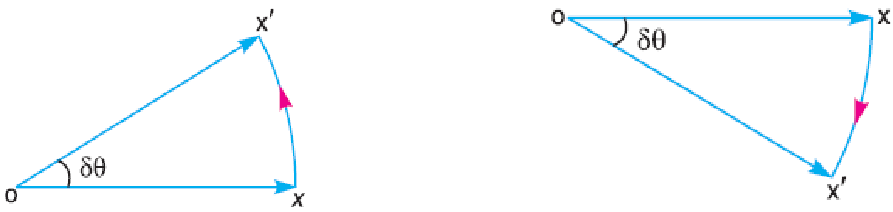


Effect of Gyroscopic Couple on a Naval Ship during Steering

Steering is the turning of a complete ship in a curve towards left or right, while it moves forward. Consider the ship taking a left turn, and rotor rotates in the clockwise direction when viewed from the stern, as shown in Fig. The effect of gyroscopic couple on a naval ship during steering taking left or right turn may be obtained in the similar way as for an aeroplane.



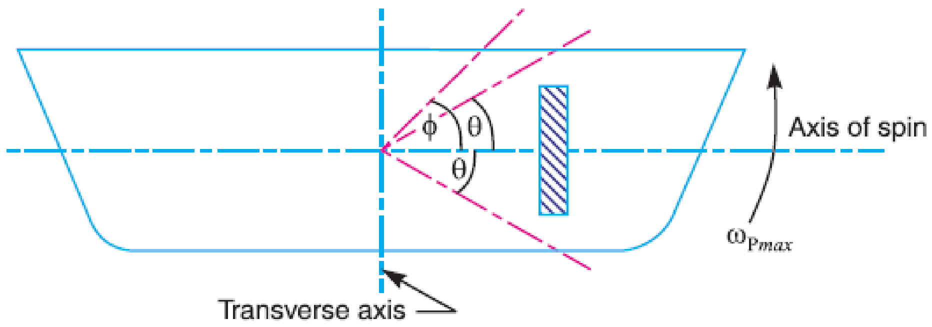
When the rotor of the ship rotates in the clockwise direction when viewed from the stern, it will have its angular momentum vector in the direction ox as shown in Fig(a). As the ship steers to the left, the active gyroscopic couple will change the angular momentum vector from ox to ox′. The vector xx′ now represents the active gyroscopic couple and is perpendicular to ox. Thus the plane of active gyroscopic couple is perpendicular to xx′ and its direction in the axis OZ for left hand turn is clockwise as shown in Fig. The reactive gyroscopic couple of the same magnitude will act in the opposite direction (i.e. in anticlockwise direction). The effect of this reactive gyroscopic couple is to raise the bow and lower the stern.



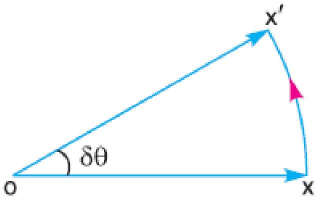
(*a*) Streeing to the left (*b*) Streeing to the right

Effect of Gyroscopic Couple on a Naval Ship during Pitching

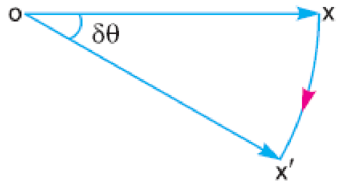
Pitching is the movement of a complete ship up and down in a vertical plane about transverse axis, as shown in Fig(*a*). In this case, the transverse axis is the axis of precession. The pitching of the ship is assumed to take place with simple harmonic motion *i.e.* the motion of the axis of spin about transverse axis is simple harmonic.



(*a*) Pitching of a naval ship



(*b*) Pitching upward



(*c*) Pitching downward

∴ Angular displacement of the axis of spin from mean position after time *t* seconds,

$\theta = \phi \sin \omega_1 . t$

ϕ = Amplitude of swing *i.e.* maximum angle turned from the mean position in radians, and

ω_1 = Angular velocity of S.H.M.

$$= \frac{2\pi}{\text{Time period of S.H.M. in seconds}} = \frac{2\pi}{t_p} \text{ rad/s}$$

Angular velocity of precession,

$$\omega_P = \frac{d\theta}{dt} = \frac{d}{dt}(\phi \sin \omega_1 . t) = \phi \omega_1 \cos \omega_1 t$$

The angular velocity of precession will be maximum, if $\cos \omega_1 . t = 1$.

∴ Maximum angular velocity of precession,

$$\omega_{Pmax} = \phi . \omega_1 = \phi \times 2\pi / t_p \qquad \text{...(Substituting } \cos \omega_1 . t = 1)$$

Let I = Moment of inertia of the rotor in kg-m^2, and
ω= Angular velocity of the rotor in rad/s.
Mamimum gyroscopic couple,

$$C_{max} = I . \omega . \omega_{Pmax}$$

When the pitching is upward, the effect of the reactive gyroscopic couple, as shown in Fig.(*b*), will try to move the ship toward star-board. On the other hand, if the pitching is downward, the effect of the reactive gyroscopic couple, as shown in Fig(*c*), is to turn the ship towards port side.

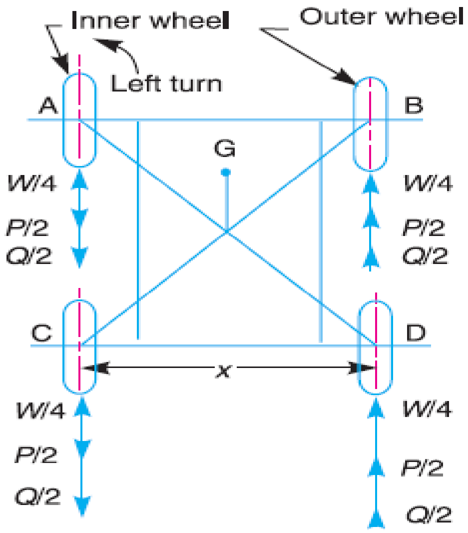
Effect of Gyroscopic Couple on a Naval Ship during Rolling

We know that, for the effect of gyroscopic couple to occur, the axis of precession should always be perpendicular to the axis of spin. If, however, the axis of precession becomes parallel to the axis of spin, there will be no effect of the gyroscopic couple acting on the body of the ship.

In case of rolling of a ship, the axis of precession (*i.e.* longitudinal axis) is always parallel to the axis of spin for all positions. Hence, there is no effect of the gyroscopic couple acting on the body of a ship.

Stability of a Four Wheel Drive Moving in a Curved Path

Consider the four wheels *A*, *B*, *C* and *D* of an automobile locomotive taking a turn towards left as shown in Fig. The wheels *A* and *C* are inner wheels, whereas *B* and *D* are outer wheels. The centre of gravity (*C.G.*) of the vehicle lies vertically above the road surface.



Let m = Mass of the vehicle in kg,

W = Weight of the vehicle in newtons = $m.g$,

r_W = Radius of the wheels in metres,

R = Radius of curvature in metres ($R > r_W$),

h = Distance of centre of gravity, vertically above the road surface in metres,

x = Width of track in metres,

I_W = Mass moment of inertia of one of the wheels in kg-m^2,

ω_W = Angular velocity of the wheels or velocity of spin in rad/s,

I_E = Mass moment of inertia of the rotating parts of the engine in kg-m2,

ω_E = Angular velocity of the rotating parts of the engine in rad/s,

G = Gear ratio $=\omega_E/\omega_w$

v = Linear velocity of the vehicle in m/s = $\omega_W.r_W$

A little consideration will show, that the weight of the vehicle (W) will be equally distributed over the four wheels which will act downwards. The reaction between each wheel and the road surface of the same magnitude will act upwards.

Therefore

Road reaction over each wheel

$= W/4 = m.g/4$ newtons

Let us now consider the effect of the gyroscopic couple and centrifugal couple on the vehicle.

1. Effect of the gyroscopic couple

Since the vehicle takes a turn towards left due to the precession and other rotating parts, therefore a gyroscopic couple will act.
We know that velocity of precession,
$\omega_P = v/R$
Gyroscopic couple due to 4 wheels,
$C_W = 4\ I_W.\omega_W.\omega_P$
and gyroscopic couple due to the rotating parts of the engine,
$C_E = I_E.\omega_E.\omega_P = I_E.G.\omega_W.\omega_P$
Net gyroscopic couple,
$C = C_W \pm C_E = 4\ I_W.\omega_W.\omega_P \pm I_E.G.\omega_W.\omega_P$
$= \omega_W.\omega_P\ (4\ I_W \pm G.I_E)$

The ***positive*** sign is used when the wheels and rotating parts of the engine rotate in the same direction. If the rotating parts of the engine revolves in opposite direction, then ***negative*** sign is used.

Due to the gyroscopic couple, vertical reaction on the road surface will be produced. The reaction will be vertically upwards on the outer wheels and vertically downwards on the inner wheels.
Let the magnitude of this reaction at the two outer or inner wheels be P newtons. Then

$$P \times x = C \text{ or } P = C/x$$

Vertical reaction at each of the outer or inner wheels,

$$P/2 = C/2x$$

2. Effect of the centrifugal couple

Since the vehicle moves along a curved path, therefore centrifugal force will act outwardly at the centre of gravity of the vehicle. The effect of this centrifugal force is also to overturn the vehicle.
We know that centrifugal force,

$$F_C = \frac{m \times v^2}{R}$$

The couple tending to overturn the vehicle or overturning couple,

$$C_O = F_C \times h = \frac{m.v^2}{R} \times h$$

This overturning couple is balanced by vertical reactions, which are vertically upwards on the outer wheels and vertically downwards on the inner wheels. Let the magnitude of this reaction at the two outer or inner wheels be Q. Then

$$Q \times x = C_O \quad \text{or} \quad Q = \frac{C_O}{x} = \frac{m.v^2.h}{R.x}$$

Vertical reaction at each of the outer or inner wheels,

$$\frac{Q}{2} = \frac{m.v^2.h}{2R.x}$$

Total vertical reaction at each of the outer wheel,

$$P_O = \frac{W}{4} + \frac{P}{2} + \frac{Q}{2}$$

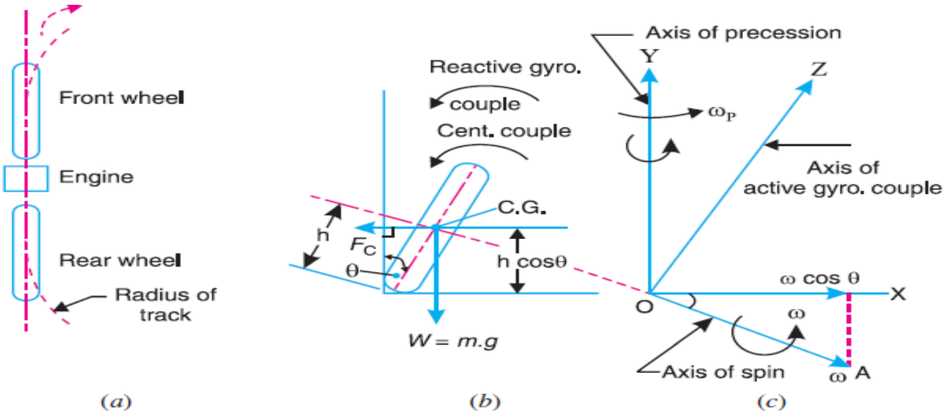
total vertical reaction at each of the inner wheel

$$P_I = \frac{W}{4} - \frac{P}{2} - \frac{Q}{2}$$

A little consideration will show that when the vehicle is running at high speeds, P_I may be zero or even negative. This will cause the inner wheels to leave the ground thus tending to overturn the automobile. In order to have the contact between the inner wheels and the ground, the sum of $P/2$ and $Q/2$ must be less than $W/4$.

Stability of a Two Wheel Vehicle Taking a Turn

Consider a two wheel vehicle (say a scooter or motor cycle) taking a right turn as shown in fig.



Let m = Mass of the vehicle and its rider in kg,

W = Weight of the vehicle and its rider in newtons = $m.g$,

h = Height of the centre of gravity of the vehicle and rider,

r_W = Radius of the wheels,

R = Radius of track or curvature,

I_W = Mass moment of inertia of each wheel,

I_E = Mass moment of inertia of the rotating parts of the v engine,

ω_W = Angular velocity of the wheels,

ω_E = Angular velocity of the engine,

G = Gear ratio = ω_E / ω_W,

v = Linear velocity of the vehicle = $\omega_W \times r_W$,

θ = Angle of heel. It is inclination of the vehicle to the vertical for equilibrium.

Let us now consider the effect of the gyroscopic couple and centrifugal couple on the vehicle,

1. **Effect of gyroscopic couple**

We know that $v = \omega_W \times r_W$ or $\omega_W = v / r_W$

$$\omega_E = G.\omega_W = G \times \frac{v}{r_W}$$

$\therefore$ Total $(I \times \omega) = 2\, I_W \times \omega_W \pm I_E \times \omega_E$

$$= 2\, I_W \times \frac{v}{r_W} \pm I_E \times G \times \frac{v}{r_W} = \frac{v}{r_W}(2\, I_W \pm G.I_E)$$

velocity of precession, $\omega_P = v / R$

A little consideration will show that when the wheels move over the curved path, the vehicle is always inclined at an angle θ with the vertical plane as shown in Fig(*b*). This angle is known as ***angle of heel***. In other words, the axis of spin is inclined to the horizontal at an angle θ, as shown in Fig (*c*). Thus the angular momentum vector *I*ω due to spin is represented by *OA* inclined to *OX* at an angle θ. But the precession axis is vertical. Therefore the spin vector is resolved along *OX*.

Gyroscopic couple,

$$C_1 = I.\omega \cos\theta \times \omega_P = \frac{v}{r_W}(2\, I_W \pm G.I_E)\cos\theta \times \frac{v}{R}$$

$$= \frac{v^2}{R.r_W}(2\, I_W \pm G.I_E)\cos\theta$$

2. Effect of centrifugal couple

We know that centrifugal force,

$$F_C = \frac{m.v^2}{R}$$

This force acts horizontally through the centre of gravity (*C.G.*) along the outward direction.

Centrifugal couple,

$$C_2 = F_C \times h\cos\theta = \left(\frac{m.v^2}{R}\right) h\cos\theta$$

Since the centrifugal couple has a tendency to overturn the vehicle, therefore

Total overturning couple,

$$C_O = \text{Gyroscopic couple} + \text{Centrifugal couple}$$

$$= \frac{v^2}{R.r_W}(2\, I_W + G.I_E)\cos\theta + \frac{m.v^2}{R} \times h\cos\theta$$

$$= \frac{v^2}{R}\left[\frac{2\, I_W + G.I_E}{r_W} + m.h\right]\cos\theta$$

We know that balancing couple = $m.g.h \sin\theta$

The balancing couple acts in clockwise direction when seen from the front of the vehicle. Therefore for stability, the overturning couple must be equal to the balancing couple, *i.e.*

$$\frac{v^2}{R}\left(\frac{2\,I_W + G.I_E}{r_W} + m.h\right)\cos\theta = m.g.h\sin\theta$$

From this expression, the value of the angle of heel (θ) may be determined, so that the vehicle does not skid.

PROBLEMS

Example 1. A uniform disc of diameter 300 mm and of mass 5 kg is mounted on one end of an arm of length 600 mm. The other end of the arm is free to rotate in a universal bearing. If the disc rotates about the arm with a speed of 300 r.p.m. clockwise, looking from the front, with what speed will it precess about the vertical axis?

Solution. Given: d = 300 mm or r = 150 mm = 0.15 m ; m = 5 kg ; l = 600 mm = 0.6 m ;N = 300 r.p.m. or $\omega = 2\pi \times 300/60 = 31.42$ rad/s

We know that the mass moment of inertia of the disc, about an axis through its centre of gravity and perpendicular to the plane of disc,

$I = m.r^{2/}2 = 5(0.15)2/2 = 0.056$ kg-m^2

couple due to mass of disc,

$C = m.g.l = 5 \times 9.81 \times 0.6 = 29.43$ N-m

Let ω_P = Speed of precession.

We know that couple (C),

$$29.43 = I.\omega.\omega_P = 0.056 \times 31.42 \times \omega_P = 1.76\ \omega_P$$

$$\omega_P = 29.43/1.76 = 16.7 \text{ rad/s}$$

Example 2. An aeroplane makes a complete half circle of 50 metres radius, towards left, when flying at 200 km per hr. The rotary engine and the propeller of the plane has a mass of 400 k gand a radius of gyration of 0.3 m. The engine rotates at 2400 r.p.m. clockwise when viewed from the rear. Find the gyroscopic couple on the aircraft and state its effect on it.

Solution. Given : R = 50 m ; v = 200 km/hr = 55.6 m/s ; m = 400 kg ; k = 0.3 m ; N = 2400 r.p.m. or $\omega = 2\pi \times 2400/60 = 251$ rad/s

We know that mass moment of inertia of the engine and the propeller,
$I = m.k^2 = 400(0.3)2 = 36$ kg-m^2

angular velocity of precession,
$\omega_P = v/R = 55.6/50 = 1.11$ rad/s
We know that gyroscopic couple acting on the aircraft,
$C = I.\ \omega.\ \omega_P = 36 \times 251.4 \times 1.11 = 100\ 46$ N-m
10.046 kN-m
when the aeroplane turns towards left, the effect of the gyroscopic couple is to lift the nose upwards and tail downwards.

Example3. The turbine rotor of a ship has a mass of 8 tonnes and a radius of gyration0.6 m. It rotates at 1800 r.p.m. clockwise, when looking from the stern. Determine the gyroscopic couple, if the ship travels at 100 km/hr and steer to the left in a curve of 75 m radius.

Solution. Given: $m = 8$ t = 8000 kg ; $k = 0.6$ m ; $N = 1800$ r.p.m. or $\omega = 2\pi \times 1800/60 = 188.5$ rad/s ; $v = 100$ km/h = 27.8 m/s ; $R = 75$ m

We know that mass moment of inertia of the rotor,

$I = m.k^2 = 8000\ (0.6)2 = 2880$ kg-m^2

angular velocity of precession,

$\omega_P = v\ /\ R = 27.8\ /\ 75 = 0.37$ rad/s

We know that gyroscopic couple,
$C = I.\omega.\omega_P = 2880 \times 188.5 \times 0.37 = 200\ 866$ N-m
= 200.866 kN-m

when the rotor rotates in clockwise direction when looking from the stern and the ship steers to the left, the effect of the reactive gyroscopic couple is to raise the bow and lower the stern.

UNIT-II

STATIC AND DYNAMIC FORCE ANALYSIS OF PLANAR MECHANISMS

Static Force Analysis

A machine is a device that performs work and, as such, transmits energy by means mechanical force from a power source to a driven load. It is necessary in the design machine mechanisms to know the manner in which forces are transmitted from input to the output, so that the components of the machine can be properly size withstand the stresses that are developed. If the members are not designed to strong enough, then failure will occur during machine operation; if, on the other hand, the machine is over designed to have much more strength than required, then the machine may not be competitive with others in terms of cost, weight, size, power requirements, or other criteria. The bucket load and static weight loads may far exceed any dynamic loads due to accelerating masses, and a static-force analysis would be justified. An analysis that includes inertia effects is called a dynamic-force analysis and will be discussed in the next chapter. An example of an application where a dynamic-force analysis would be required is in the design of an automatic sewing machine, where, due to high operating speeds, the inertia forces may be greater than the external loads on the machine.

Another assumption deals with the rigidity of the machine components. No material is truly rigid, and all materials will experience significant deformation if the forces, either external or inertial in nature, are great enough. It will be assumed in this chapter and the next that deformations are so small as to be negligible and, therefore, the members will be treated as though they are rigid. The subject of mechanical vibrations, which is beyond the scope of this book, considers the flexibility of machine components and the resulting effects on machine behavior. A third major assumption that is often made is that friction effects are negligible. Friction is inherent in all devices, and its degree is dependent upon many factors, including types of bearings, lubrication, loads, environmental conditions, and so on. Friction will be neglected in the first few sections of this chapter, with an introduction to the subject presented. In addition to assumptions of the types discussed above, other assumptions may be necessary, and some of these will be addressed at various points throughout the chapter.

The first part of this chapter is a review of general force analysis principles and will also establish some of the convention and terminology to be used in succeeding sections. The remainder of the chapter will then present both graphical and analytical methods for static-force analysis of machines.

Free-Body Diagrams

Engineering experience has demonstrated the importance and usefulness of free-body diagrams in force analysis. A free-body diagram is a sketch or drawing of part or all of a system, isolated in order to determine the nature of forces acting on that body. Sometimes a free-body diagram may take the form of a mental picture; however, actual sketches are strongly recommended, especially for complex mechanical systems.

Generally, the first, and one of the most important, steps in a successful force analysis is the identification of the free bodies to be used. Figures 5.1B through 5.1E show examples of various free bodies that might be considered in the analysis of the four-bar linkage shown in Figure 5.1A. In Figure 5.1B, the free body consists of the three moving members isolated from the frame; here, the forces acting on the free body include a driving force or torque, external loads, and the forces transmitted:

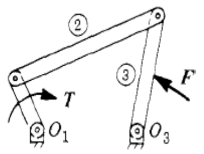


Figure 5.1(A) A four-bar linkage.

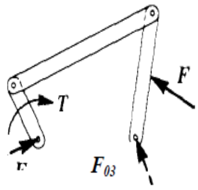


Figure 5.1(B) Free-body diagram of the three moving links

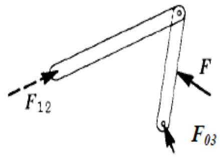


Figure 5.1(C) Free-body diagram of two connected links

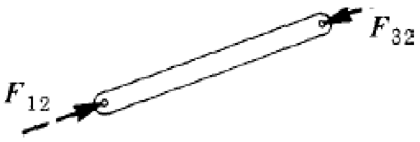


Figure 5.1(D) Free-body diagram of a single link

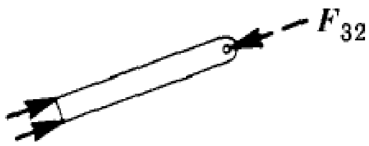


Figure 5.1(E) Free body diagram of part of a link.

Static Equilibrium

For a free body in static equilibrium, the vector sum of all forces acting on the body must be zero and the vector sum of all moments about any arbitrary point must also be zero. These conditions can be expressed mathematically as follows:

$$\sum F = 0 \qquad (5.1A)$$

$$\sum T = 0 \qquad (5.1B)$$

Since each of these vector equations represents three scalar equations, there are a total of six independent scalar conditions that must be satisfied for the general case of equilibrium under three-dimensional loading.

There are many situations where the loading is essentially planar; in which case, forces can be described by two-dimensional vectors. If the *xy* plane designates the plane of loading, then the applicable form of Eqs. 5.1A and 5.1B is:-

$$\sum F_x = 0 \qquad (5.2A)$$

$$\sum F_y = 0 \qquad (5.2B)$$

$$\sum T_z = 0 \qquad (5.2C)$$

Eqs. 5.2A to 5.2C are three scalar equations that state that, for the case of two-dimensional *xy* loading, the summations of forces in the *x* and *y* directions must individually equal zero and the summation of moments about any arbitrary point in the plane must also equal zero. The remainder of this chapter deals with two-dimensional force analysis. A common example of three-dimensional forces is gear forces.

Graphical Force Analysis:

Graphical force analysis employs scaled free-body diagrams and vector graphics in the determination of unknown machine forces. The graphical approach is best suited for planar force systems. Since forces are normally not constant during machine motion. Analyses may be required for a number of mechanism positions; however, in many cases, critical maximum-force positions can be identified and graphical analyses performed for these positions only. An important advantage of the graphical approach is that it provides useful insight as to the nature of the forces in the physical system.

This approach suffers from disadvantages related to accuracy and time. As is true of any graphical procedure, the results are susceptible to drawing and measurement errors. Further, a great amount of graphics time and effort can be expended in the iterative design of a machine mechanism for which fairly thorough knowledge of force-time relationships is required. In recent years, the physical insight of the graphics approach and the speed and accuracy inherent in the computer-based analytical approach have been brought together through computer graphics systems, which have proven to be very effective engineering design tools. There are a few special types of member loadings that are repeatedly encountered in the force analysis of mechanisms, These include a member subjected to two forces, a member subjected to three forces, and a member subjected to two forces and a couple. These special cases will be considered in the following paragraphs, before proceeding to the graphical analysis of complete mechanisms

Analysis of a Two-Force Member:

A *member subjected to two forces is in equilibrium if and only if the two forces (1) have the same magnitude, (2) act along the same line, and (3) are opposite in sense.* Figure 5.2A shows a free-body diagram of a member acted upon by forcesF_1 and F_2 where the points of application of these forces are points A and B. For equilibrium the directions of F_1 and F_2 must be along line AB and F_1 must equal $-F_2$ graphical vector addition of forces F_1 and F_2 is shown in Figure 5.2B, and, obviously, the resultant net force on the member is zero when $F_1 = -F_2$. The resultant moment about any point will also be zero.

Thus, if the load application points for a two-force member are known, the line of action of the forces is defined, and it the magnitude and sense of one of the forces are known, then the other

Force can immediately be determined. Such a member will either be in tension or compression.

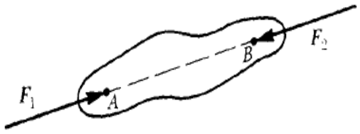


Figure 5.2(A) A two-force member. The resultant force and the resultant moment both equal Zero.

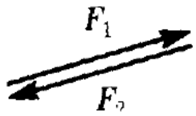


Figure 5.2(B) Force summation for a two-force member

Analysis of a Three-Force Member

A member subjected to three forces is in equilibrium if and only if (1) the resultant of the three forces is zero, and (2) the lines of action of the forces all intersect at the same point. The first condition guarantees equilibrium of forces, while the second condition guarantees equilibrium of moments. The second condition can be under-stood by considering the case when it is not satisfied. See Figure 5.3A. If moments are summed about point P, the intersection of forces F_1 and F_2 , then the moments of these forces will be zero, but F_3 will produce a nonzero moment, resulting in a nonzero net moment on the member. On the other hand, if the line of action of force F_3 also passes through point P (Figure 5.3B), the net moment will be zero. This common point of intersection of the three forces is called the point of concurrency.

A typical situation encountered is that when one of the forces, F_1 , is known completely, magnitude and direction, a second force, F_2 , has known direction but unknown magnitude, and force F_3 has unknown magnitude and direction. The graphical solution of this case is depicted in Figures 5.4A through 5.4C. First, the free-body diagram is drawn to a convenient scale and the points of application of the three forces are identified. These are points *A*, *B*, and *C*. Next, the known force F_1 is drawn on the diagram with the proper direction and a suitable magnitude scale. The direction of force F_2 is then drawn, and the intersection of this line with an extension of the line of action of force F_1 is the concurrency point *P*. For equilibrium, the line of action of force F_3 must pass through points *C* and *P* and is therefore as shown in Figure 5.4A.

The force equilibrium condition states that

$$F_1 + F_2 + F_3 = 0$$

Since the directions of all three forces are now known and the magnitude of F_1 were given, this equation can be solved for the remaining two magnitudes. A graphical Solution follows from the fact that the three forces must form a closed vector loop, called a force polygon. The procedure is shown in Figure 5.4B. Vector F_1 is redrawn

From the head of this vector, a line is drawn in the direction of force F_2, and from the tail, a line is drawn parallel to F_3 . The intersection of these lines closes the vector loop and determines the magnitudes of forces F_2 and F_3. Note that the same solution is obtained if, instead, a line parallel to F_3 is drawn from the head of F_1 and a line parallel to F_2 is drawn from the tail of F_1 . See Figure 5.4C

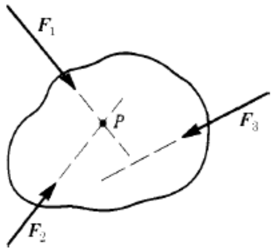


Figure 5.3(B) The three forces intersect at the same point *P*, called the *concurrency point,* and the net moment is zero.

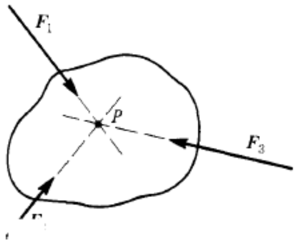


Figure 5.3(A) The three forces on the member do not intersect at a common point and there is a nonzero resultant moment.

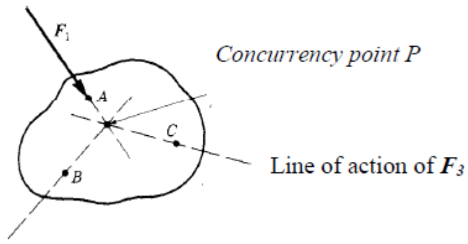


Figure 5.4(A) Graphical force analysis of a three- force member.

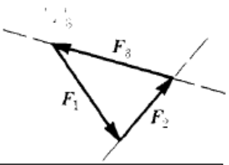


Figure 5.4(B) Force polygon for the three forces member.

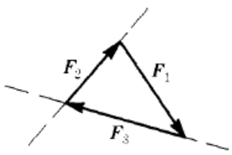


Figure 5.4(C) An equivalent force polygon for the three force member

This is so because vector addition is commutative, and, therefore, both force polygons are equivalent to the vector equation above. It is important to remember that, by the definition of vector addition, the force polygon corresponding to the general force equation

$$\sum F = 0$$

Will have adjacent vectors connected head to tail. This principle is used in identifying the sense of forces F_2 and F_3 in Figures 5.4B and 5.4C. Also, if the lines of action ofF_1 and F_2 are parallel," then the point of concurrency is at infinity, and the third force F_3 must be parallel to the other two. In this case, the force polygon collapses to a straight line.

Dynamic Force Analysis

D'Alembert's Principle and Inertia Forces

An important principle, known as d' Alembert's principle, can be derived from Newton's second law. In words, d' Alembert's principle states that the reverse-effective forces and torques and the external forces and torques on a body together give statical equilibrium

$$F + (-ma_G) = 0 \qquad (5.3A)$$

$$T_{eG} + (-I_G\alpha) = 0 \qquad (5.3B)$$

The terms in parentheses in Eqs. 5.3A and 5.3B are called the reverse-effective force and the reverse-effective torque, respectively. These quantities are also referred to as inertia force and inertia torque. Thus, we define the inertia force ***F***, as

$$\boldsymbol{F_i = -ma_G} \qquad (5.4A)$$

This reflects the fact that a body resists any change in its velocity by an inertia force proportional to the mass of the body and its acceleration. The inertia force acts through the center of mass ***G*** of the body. The inertia torque or inertia couple ***C***, is given by:

$$C_i = -I_G\alpha \qquad (5.4B)$$

As indicated, the inertia torque is a pure torque or couple. From Eqs. 5.4A and 5.4B, their directions are opposite to that of the accelerations. Substitution of Eqs. 5.4A and 5.4B into Eqs, 5.3A and 5.3B leads to equations that are similar to those used for static-force analysis:

$$\sum F = \sum F_e + F_i = 0 \qquad (5.5A)$$

$$\sum T_G = \sum T_{eG} + C_i = 0 \qquad (5.5B)$$

Where $\sum F$ refers here to the summation of external forces and, therefore, is the resultant external force, and $\sum T_{eG}$ is the summation of external moments, or resultant external moment, about the center of mass ***G***. Thus, the dynamic analysis problem is reduced in form to a static force and moment balance where inertia effects are treated in the same manner as external forces and torques. In particular for the case of assumed mechanism motion, the inertia forces and couples can be determined completely and thereafter treated as known mechanism loads.

Furthermore, d' Alembert's principle facilitates moment summation about any arbitrary point *P* in the body, if we remember that the moment due to inertia force ***F,*** must be included in the summation. Hence,

$$\sum T_P = \sum T_{eP} + C_i + R_{PG} \times F_i = 0 \qquad (5.5C)$$

Where; $\sum T_P$ is the summation of moments, including inertia moments, about point

$\sum T_{eP}$ is the summation of external moments about ***P***, ***C***, is the inertia couple defined by Eq. 5.4B, ***F***, is the inertia force defined by Eq. 5.4A, and ***RPG*** is a vector from point P to point ***C***. It is clear that Eq. 5.5B is the special case of Eq.5.5C, where point ***P*** is taken as the center of mass G (i.e., ***RPG*** =0).

For a body in plane motion in the ***x y*** plane with all external forces in that plane. Eqs. 5.5A and 5.5B become:

$$\sum F_x = \sum F_{ex} + F_{ix} = \sum F_{ex} + (-ma_{Gx}) = 0 \qquad (5.6A)$$

$$\sum F_y = \sum F_{ey} + F_{iy} = \sum F_{ey} + (-ma_{Gy}) = 0 \qquad (5.6B)$$

$$\sum T_G = \sum T_{eG} + C_i = \sum T_{eG} + (-I_G \alpha) = 0 \qquad (5.6C)$$

Where ***aGx*** and ***aGy*** are the ***x*** and ***y*** components of ***aG***. These are three scalar equations, where the sign convention for torques and angular accelerations is based on a right-hand ***xyz*** coordinate system; that is. Counterclockwise is positive and clockwise is negative. The general moment summation about arbitrary point ***P***, Eq. 5.5C, becomes:

$$\begin{aligned} \sum T_P &= \sum T_{eP} + C_i + R_{PGx} . F_{iy} - R_{PGy} . F_{ix} \\ &= \sum T_{eP} + (-I_G \alpha) + R_{PGx} (-ma_{Gy}) - R_{PGy} (-ma_{Gx}) = 0 \end{aligned} \qquad (5.6D)$$

Where ***RPGx*** and ***RPGy*** are the ***x*** and ***y*** components of position vector ***R****PG*. This expression for dynamic moment equilibrium will be useful in the analyses to be presented in the following sections of this chapter.

Equivalent Offset Inertia Force

For purposes of graphical plane force analysis, it is convenient to define what is known as the equivalent offset inertia force. This is a single force that accounts for both translational inertia and rotational inertia corresponding to the plane motion of a rigid body. Its derivation will follow, with reference to Figures 5.7A through 5.7D.

Figure 5.7A shows a rigid body with planar motion represented by center of mass acceleration ***aC*** and angular acceleration α . The inertia force and inertia torque associated with this motion are also shown. The inertia torque $-I_G\,\alpha$ can be expressed as a couple consisting of forces ***Q*** and (- ***Q***) separated by perpendicular

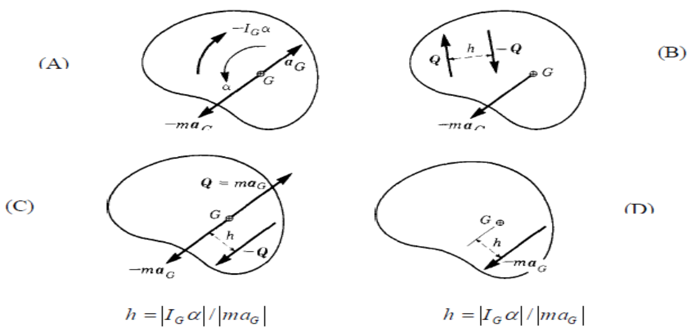


Figure 5.7 (A) Derivation of the equivalent offset inertia force associated with planer motion of a rigid body. (B) Replacement of the inertia torque by a couple. (C) The strategic choice of a couple. (D) The single force is equivalent to the combination of a force and a torque in figure 5.7(A)

Distance ***h*** , as shown in Figure 5.7B. The necessary conditions for the couple to be equivalent to the inertia torque are that the sense and magnitude be the same. Therefore, in this case, the sense of the couple must be clockwise and the magnitudes of ***Q*** and ***h*** must satisfy the relationship

$$| \; Q.h| = | \; I_G .\alpha|$$

Otherwise, the couple is arbitrary and there are an infinite number of possibilities that will work. Furthermore, the couple can be placed anywhere in the plane.

Figure 5.7C shows a special case of the couple, where force vector ***Q*** is equal to $\boldsymbol{ma_G}$ and acts through the center of mass. Force ***(- Q)*** must then be placed as shown to produce a clockwise sense and at a distance;

$$h = \frac{| \; I_G \alpha|}{| \; Q|} = \frac{| \; I_G \alpha|}{|ma_G|}$$

Force ***Q*** will cancel with the inertia force $\boldsymbol{F_i = - ma_G}$, leaving the single equivalent offset force shown in Figure 5.7D, which has the following characteristics:

- The magnitude of the force is | $\boldsymbol{ma_G}$ |.
- The direction of the force is opposite to that of acceleration α .
- The perpendicular offset distance from the center of mass to the line of action of the force is given by Eq. 5.7.
- The force is offset from the center of mass so as to produce a moment about the center of mass that is opposite in sense to acceleration *a*.

The usefulness of this approach for graphical force analysis will be demonstrated in the following section. It should be emphasized, however, that this approach is usually unnecessary in analytical solutions, where Eqs. 5.6A to 5.6D. Including the original inertia force and inertia torque, can be applied directly

FRICTION IN MACHINE ELEMENTS

Screw Friction

The screws, bolts, studs, nuts etc. are widely used in various machines and structures for temporary fastenings. These fastenings have screw threads, which are made by cutting a continuous helical groove on a cylindrical surface. If the threads are cut on the outer surface of a solid rod, these are known as ***external threads***. But if the threads are cut on the internal surface of a hollow rod, these are known as ***internal threads***. The screw threads are mainly of two types *i.e.* V-threads and square threads. The V-threads are stronger and offer more frictional resistance to motion than square threads. Moreover, the V-threads have an advantage of preventing the nut from slackening. In general, the V threads are used for the purpose of tightening pieces together *e.g.* bolts and nuts etc. But the square threads are used in screw jacks, vice screws etc. The following terms are important for the study of screw

1. ***Helix.*** It is the curve traced by a particle, while describing a circular path at a uniform speed and advancing in the axial direction at a uniform rate. In other words, it is the curve traced by a particle while moving along a screw thread.

2. ***Pitch***. It is the distance from a point of a screw to a corresponding point on the next thread, measured parallel to the axis of the screw

3. ***Lead***. It is the distance; a screw thread advances axially in one turn.

4. ***Depth of thread***. It is the distance between the top and bottom surfaces of a thread (also known as **crest** and **root** of a thread).

5. **Single-threaded screw**. If the lead of a screw is equal to its pitch, it is known as single threaded screw.

   Lead = Pitch × Number of threads

6. ***Helix angle.*** It is the slope or inclination of the thread with the horizontal.

$$\tan\alpha = \frac{\text{Lead of screw}}{\text{Circumference of screw}}$$

$$= p/\pi d \qquad \text{...(In single-threaded screw)}$$

$$= n.p/\pi d \qquad \text{...(In multi-threaded screw)}$$

α = Helix angle,

p = Pitch of the screw,

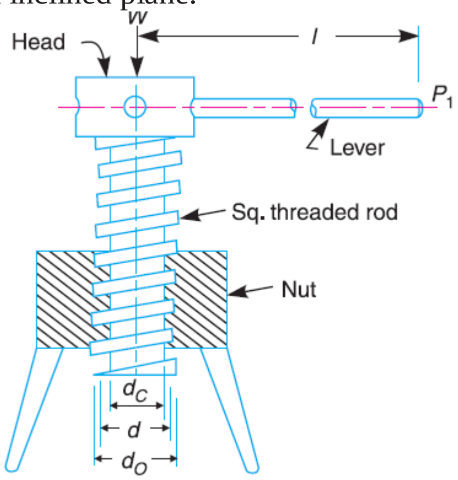
d = Mean diameter of the screw, and

n = Number of threads in one lead.

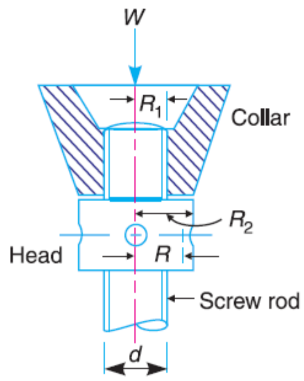
Screw Jack.

Screw Jack

The screw jack is a device, for lifting heavy loads, by applying a comparatively smaller effort at its handle. The principle, on which a screw jack works, is similar to that of an inclined plane.



(*a*) Screw jack.

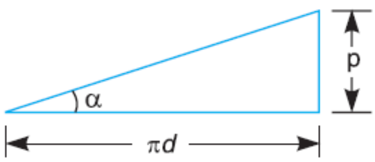


(*b*) Thrust collar.

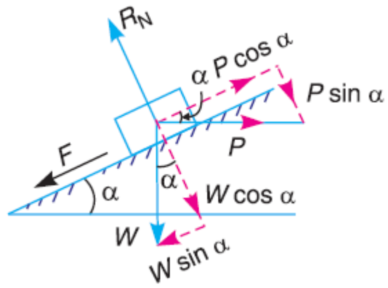
Fig (*a*) shows a common form of a screw jack, which consists of a square threaded rod (also called screw rod or simply screw) which fits into the inner threads of the nut. The load, to be raised or lowered, is placed on the head of the square threaded rod which is rotated by the application of an effort at the end of the lever for lifting or lowering the load.

Torque Required to Lift the Load by a Screw Jack

If one complete turn of a screw thread by imagined to be unwound, from the body of the screw and developed, it will form an inclined plane as shown in Fig(*a*).



(*a*) Development of a screw.



(*b*) Forces acting on the screw.

Let p = Pitch of the screw,

d = Mean diameter of the screw,

α= Helix angle,

P = Effort applied at the circumference of the screw to lift the load,

W = Load to be lifted, and

μ = Coefficient of friction, between the screw and nut = $\tan \phi$,

Where φ is the friction angle.

From the geometry of the Fig(*a*), we find that

$$\tan \alpha = p/\pi d$$

Since the principle on which a screw jack works is similar to that of an inclined plane, therefore the force applied on the lever of a screw jack may be considered to be horizontal as shown in Fig(*b*).

Since the load is being lifted, therefore the force of friction ($F = \mu.R_N$) will act downwards. All the forces acting on the screw are shown in Fig(*b*).

Resolving the forces along the plane,

$P \cos \alpha = W \sin \alpha + F = W \sin\alpha + \mu.R_N$ ....***(i)***

and resolving the forces perpendicular to the plane,

$R_N = P \sin \alpha + W \cos \alpha$................................***(ii)***

Substituting this value of R_N in equation (*i*),

$P \cos \alpha = W \sin \alpha + \mu (P \sin \alpha + W \cos \alpha)$

$= W \sin \alpha + \mu P \sin \alpha + \mu W \cos \alpha$

$P \cos \alpha - \mu P \sin \alpha = W \sin \alpha + \mu W \cos \alpha$

$P (\cos \alpha - \mu \sin \alpha) = W (\sin \alpha + \mu \cos \alpha)$

$$P = W \times \frac{\sin \alpha + \mu \cos \alpha}{\cos \alpha - \mu \sin \alpha}$$

Substituting the value of $\mu = \tan \varphi$ in the above equation, we get

$$P = W \times \frac{\sin \alpha + \tan \phi \cos \alpha}{\cos \alpha - \tan \phi \sin \alpha}$$

Multiplying the numerator and denominator by $\cos\varphi$,

$$P = W \times \frac{\sin \alpha \cos \phi + \sin \phi \cos \alpha}{\cos \alpha \cos \phi - \sin \alpha \sin \phi} = W \times \frac{\sin(\alpha + \phi)}{\cos(\alpha + \phi)}$$

$= W \tan (\alpha + \varphi)$

Torque required to overcome friction between the screw and nut,

$$T_1 = P \times \frac{d}{2} = W \tan(\alpha + \phi) \frac{d}{2}$$

When the axial load is taken up by a thrust collar or a flat surface, as shown in Fig (*b*),so that the load does not rotate with the screw, then the torque required to overcome friction at the collar,

$$T_2 = \mu_1.W \left(\frac{R_1 + R_2}{2} \right) = \mu_1.W.R$$

R_1 and R_2 = Outside and inside radii of the collar,
R = Mean radius of the collar, and

μ_1 = Coefficient of friction for the collar.
Total torque required to overcome friction (*i.e.* to rotate the screw),

$$T = T_1 + T_2 = P \times \frac{d}{2} + \mu_1 . W . R$$

If an effort P_1 is applied at the end of a lever of arm length l, then the total torque required to overcome friction must be equal to the torque applied at the end of the lever, *i.e.*

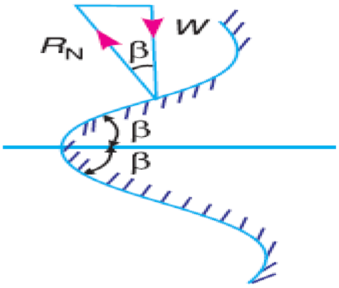
$$T = P \times \frac{d}{2} = P_1 . l$$

Friction of a V-thread

The normal reaction in case of a square threaded screw is

$R_N = W \cos\alpha$, where α= Helix angle.

But in case of V-thread (or acme or trapezoidal threads), the normal reaction between the screw and nut is increased because the axial component of this normal reaction must be equal to the axial load W, as shown in Fig.



Let 2β= Angle of the V-thread, and
β= Semi-angle of the V-thread.

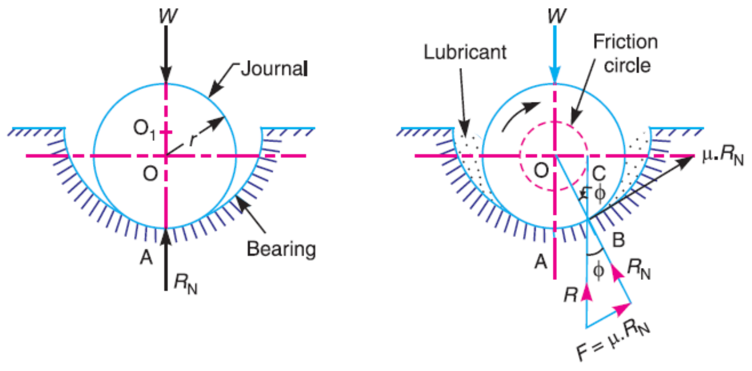
$$R_N = \frac{W}{\cos\beta}$$

frictional force, $F = \mu . R_N = \mu \times \frac{W}{\cos\beta} = \mu_1 . W$

$\frac{\mu}{\cos\beta} = \mu_1$, known as virtual coefficient of friction.

Friction in Journal Bearing-Friction Circle

A journal bearing forms a turning pair as shown in Fig(*a*). The fixed outer element of a turning pair is called a ***bearing*** and that portion of the inner element (*i.e.* shaft) which fits in the bearing is called a ***journal.*** The journal is slightly less in diameter than the bearing, in order to permit the free movement of the journal in a bearing.



(*a*) (*b*)

When the bearing is not lubricated (or the journal is stationary), then there is a line contact between the two elements as shown in Fig (a). The load W on the journal and normal reaction R_N (equal to W) of the bearing acts through the centre. The reaction RN acts vertically upwards at point A. This point A is known as **seat** or **point of pressure.**

Now consider a shaft rotating inside a bearing in clockwise direction as shown in Fig(*b*). The lubricant between the journal and bearing forms a thin layer which gives rise to a greasy friction. Therefore, the reaction *R* does not act vertically upward, but acts at another point of pressure *B*. This is due to the fact that when shaft rotates, a frictional force $F = \mu R_N$ acts at the circumference of the shaft which has a tendency to rotate the shaft in opposite direction of motion and this shifts the point *A* to point *B*.

In order that the rotation may be maintained, there must be a couple rotating the shaft.

Let ø= Angle between *R* (resultant of *F* and *R*N) and R_N,
μ = Coefficient of friction between the journal and bearing,
T = Frictional torque in N-m, and
r = Radius of the shaft in metres.

For uniform motion, the resultant force acting on the shaft must be zero and the resultant turning moment on the shaft must be zero. In other words,

$R = W$, and $T = W \times OC = W \times OB \sin\varphi = W.r\sin\varphi$
Since ø is very small, therefore substituting $\sin\varphi = \tan\varphi$

$T = W.r \tan ø = \mu.W.r$ ....($\mu = \tan ø$)

If the shaft rotates with angular velocity ω rad/s, then power wasted in friction,

$P = T\omega = T \times 2\pi N/60$ watts

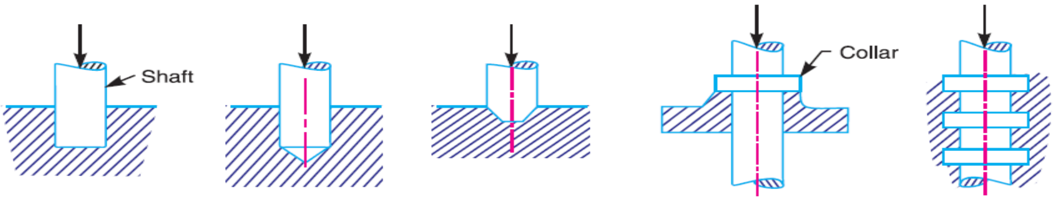
Where *N* = Speed of the shaft in r.p.m.

Friction of Pivot and Collar Bearing

The rotating shafts are frequently subjected to axial thrust. The bearing surfaces such as pivot and collar bearings are used to take this axial thrust of the rotating shaft. The propeller shafts of ships, the shafts of steam turbines, and vertical machine shafts are examples of shafts which carry an axial thrust.

The bearing surfaces placed at the end of a shaft to take the axial thrust are known as pivots. The pivot may have a flat surface or conical surface as shown in Fig. 10.16 (a) and (b) respectively. When the cone is truncated, it is then known as truncated or trapezoidal pivot as shown in Fig (c).

The collar may have flat bearing surface or conical bearing surface, but the flat surface is most commonly used. There may be a single collar, as shown in Fig (*d*) or several collars along the length of a shaft, as shown in Fig(*e*) in order to reduce the intensity of pressure.



(*a*) Flat pivot. (*b*) Conical pivot. (*c*) Truncated pivot. (*d*) Single flat collar. (*e*) Multiple flat collar.

In modern practice, ball and roller thrust bearings are used when power is being transmitted and when thrusts are large as in case of propeller shafts of ships.

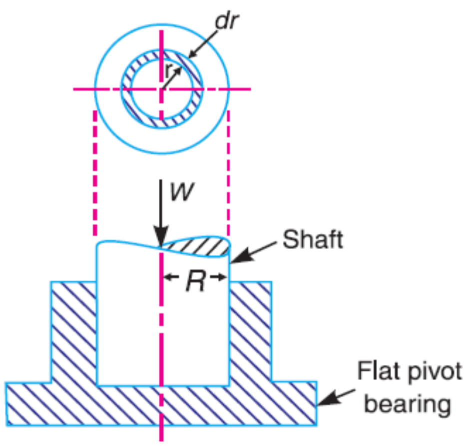
A little consideration will show that in a new bearing, the contact between the shaft and bearing may be good over the whole surface. In other words, we can say that the pressure over the rubbing surfaces is uniformly distributed. But when the bearing becomes old, all parts of the rubbing surface will not move with the same velocity, because the velocity of rubbing surface increases with the distance from the axis of the bearing. This means that wear may be different at different radii and this causes to alter the distribution of pressure. Hence, in the study of friction of bearings, it is assumed hat

1. The pressure is uniformly distributed throughout the bearing surface, and

2. The wear is uniform throughout the bearing surface.

Flat Pivot Bearing

When a vertical shaft rotates in a flat pivot bearing (known as **foot step bearing**), as shown in Fig., the sliding friction will be along the surface of contact between the shaft and the bearing.



Let W =Load transmitted over the bearing surface,
R =Radius of bearing surface,
p =Intensity of pressure per unit area of bearing Surface between rubbing surfaces, and
μ =Coefficient of friction.

We will consider the following two cases:

1. When there is a uniform pressure
2. When there is a uniform wear

1. **Considering uniform pressure**

When the pressure is uniformly distributed over the bearing area, then

$$p = \frac{W}{\pi R^2}$$

Consider a ring of radius r and thickness dr of the bearing area.
Area of bearing surface, $A = 2\pi r.dr$
Load transmitted to the ring,

$\delta W = p \times A = p \times 2\pi r.dr$ ....**(*i*)**

Frictional resistance to sliding on the ring acting tangentially at radius r,

$F_r = \mu.\delta W = \mu p \times 2\pi r.dr = 2\pi\mu.p.r.dr$

Frictional torque on the ring,

$T_r = F_r \times r = 2\pi \mu p r.dr \times r = 2\pi \mu p r^2 dr$ ……….**(*ii*)**

Integrating this equation within the limits from 0 to R for the total frictional torque on the pivot bearing.

$$\therefore \text{ Total frictional torque, } T = \int_0^R 2\pi\mu p r^2 \, dr = 2\pi\mu p \int_0^R r^2 \, dr$$

$$= 2\pi\mu p \left[\frac{r^3}{3}\right]_0^R = 2\pi\mu p \times \frac{R^3}{3} = \frac{2}{3} \times \pi\mu.p.R^3$$

$$= \frac{2}{3} \times \pi\mu \times \frac{W}{\pi R^2} \times R^3 = \frac{2}{3} \times \mu.W.R \qquad \ldots\left(\because \; p = \frac{W}{\pi R^2}\right)$$

When the shaft rotates at ω rad/s, then power lost in friction,

$$P = T.\omega = T \times 2\pi N/60 \qquad \ldots(\because \; \omega = 2\pi N/60)$$

N = Speed of shaft in r.p.m.

2. Considering uniform wear

We have already discussed that the rate of wear depends upon the intensity of pressure (p) and the velocity of rubbing surfaces (v). It is assumed that the rate of wear is proportional to the product of intensity of pressure and the velocity of rubbing surfaces (*i.e.* *p.v.*). Since the velocity of rubbing surfaces increases with the distance (*i.e.* radius r) from the axis of the bearing, therefore for uniform Wear

$$p.r = C \text{ (a constant)} \quad \text{or} \quad p = C/r$$

ınd the load transmitted to the ring,

$$\delta W = p \times 2\pi r.dr \qquad \text{...[From equation } (i)]$$

$$= \frac{C}{r} \times 2\pi r.dr = 2\pi C.dr$$

∴ Total load transmitted to the bearing

$$W = \int_0^R 2\pi\, C.dr = 2\pi\, C\,[r]_0^R = 2\pi C.R \quad \text{or} \quad C = \frac{W}{2\pi R}$$

We know that frictional torque acting on the ring,

$$T_r = 2\pi\mu\, p\, r^2\, dr = 2\pi\mu \times \frac{C}{r} \times r^2\, dr \qquad \ldots\left(\because\ p = \frac{C}{r}\right)$$

$$= 2\pi\,\mu.C.r\, dr \qquad \ldots(iii)$$

∴ Total frictional torque on the bearing,

$$T = \int_0^R 2\pi\,\mu.C.r.dr = 2\pi\mu.C\left[\frac{r^2}{2}\right]_0^R$$

$$= 2\pi\mu.C \times \frac{R^2}{2} = \pi\mu.C.R^2$$

$$= \pi\mu \times \frac{W}{2\pi R} \times R^2 = \frac{1}{2} \times \mu.W.R \qquad \ldots\left(\because\ C = \frac{W}{2\pi R}\right)$$

PROBLEMS

Example 1. A vertical shaft 150 mm in diameter rotating at 100 r.p.m. rests on a flat end foot step bearing. The shaft carries a vertical load of 20 kN. Assuming uniform pressure distribution and coefficient of friction equal to 0.05, estimate power lost in friction.

Solution. Given : D = 150 mm or R = 75 mm = 0.075 m ; N = 100 r.p.m or ω= 2 δ × 100/60= 10.47 rad/s ; W = 20 kN = 20 × 103 N ; μ = 0.05

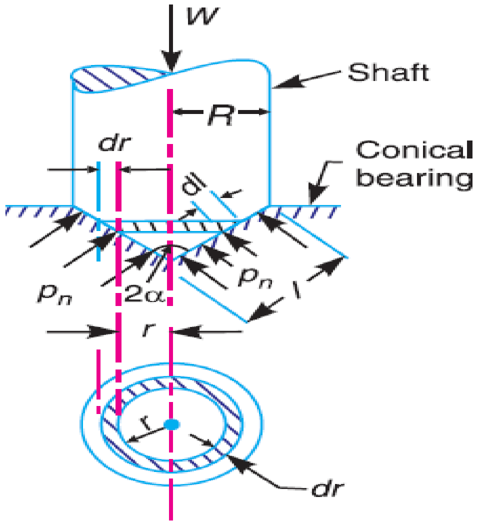
We know that for uniform pressure distribution, the total frictional torque,

$$T = \frac{2}{3} \times \mu.W.R = \frac{2}{3} \times 0.05 \times 20 \times 10^3 \times 0.075 = 50 \text{ N-m}$$

∴ Power lost in friction,

$$P = T.\omega = 50 \times 10.47 = 523.5 \text{ W} \quad \textbf{Ans.}$$

Conical Pivot Bearing



The conical pivot bearing supporting a shaft carrying a load *W* is shown in Fig.

Let P_n = Intensity of pressure normal to the cone,
α = Semi angle of the cone,
μ = Coefficient of friction between the shaft and the bearing,
R = Radius of the shaft.

Consider a small ring of radius *r* and thickness *dr*.
Let *dl* is the length of ring along the cone, such that

$dl = dr \operatorname{cosec} \alpha$

Area of the ring, $A = 2\pi r.dl = 2\pi r.dr \operatorname{cosec} \alpha$……….($dl = dr \operatorname{cosec} \alpha$)

1. ***Considering uniform pressure***

We know that normal load acting on the ring,

δW_n = Normal pressure × Area

$= p_n \times 2\pi r.dr \operatorname{cosec} \alpha$

vertical load acting on the ring,

δW = Vertical component of $\delta W_n = \delta W_n.\sin\alpha$

Total vertical load transmitted to the bearing,

$$W = \int_0^R p_n \times 2\pi r.dr = 2\pi\, p_n \left[\frac{r^2}{2}\right]_0^R = 2\pi\, p_n \times \frac{R^2}{2} = \pi R^2.p_n$$

$$p_n = W / \pi R^2$$

We know that frictional force on the ring acting tangentially at radius *r*,

$$T_r = F_r \times r = 2\pi\mu.p_n.\operatorname{cosec}\alpha.r.dr \times r = 2\pi\mu.p_n \operatorname{cosec}\alpha.r^2.dr$$

The vertical load acting on the ring is also given by
δW = Vertical component of $p_n \times$ Area of the ring

$$= p_n \sin \alpha \times 2\pi r.dr.\text{cosec}\ \alpha = p_n \times 2\pi r.dr$$

Integrating the expression within the limits from 0 to R for the total frictional torque on the conical pivot bearing.

Total frictional torque,

$$T = \int_0^R 2\pi \mu.p_n\ \text{cosec}\ \alpha.r^2\ dr = 2\pi\mu.p_n.\text{cosec}\ \alpha \left[\frac{r^3}{3}\right]_0^R$$

$$= 2\pi\mu.p_n.\text{cosec}\ \alpha \times \frac{R^3}{3} = \frac{2\pi R^3}{3} \times \mu.p_n.\text{cosec}\ \alpha \qquad \text{...}(i)$$

Substituting the value of p_n in equation (i),

$$T = \frac{2\pi R^3}{3} \times \pi \times \frac{W}{\pi R^2} \times \text{cosec}\ \alpha = \frac{2}{3} \times \mu.W.R.\ \text{cosec}\ \alpha$$

2. Considering uniform wear

In Fig. let p_r be the normal intensity of pressure at a distance r from the central axis. We know that, in case of uniform wear, the intensity of pressure varies inversely with the distance.

$p_r.r = C$ (a constant) or $p_r = C/r$

the load transmitted to the ring,

$$\delta W = p_r \times 2\pi r.dr = \frac{C}{r} \times 2\pi r.dr = 2\pi C.dr$$

Total load transmitted to the bearing,

$$W = \int_0^R 2\pi C.dr = 2\pi C\ [r]_0^R = 2\pi C.R \quad \text{or} \quad C = \frac{W}{2\pi R}$$

We know that frictional torque acting on the ring,

$$T_r = 2\pi\mu.p_r.\text{cosec}\ \alpha.r^2.dr = 2\pi\mu \times \frac{C}{r} \times \text{cosec}\,\alpha.r^2.dr$$

$$= 2\pi\mu.C.\text{cosec}\ \alpha.r.dr$$

Total frictional torque acting on the bearing,

$$T = \int_0^R 2\pi\ \mu.C.\text{cosec}\ \alpha.r.dr = 2\pi\mu.C.\text{cosec}\ \alpha \left[\frac{r^2}{2}\right]_0^R$$

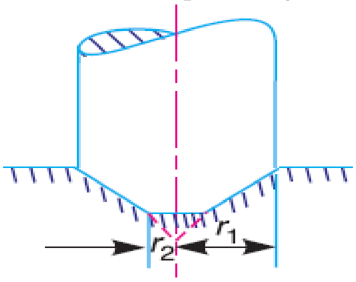
$$= 2\pi\ \mu.C.\text{cosec}\ \alpha \times \frac{R^2}{2} = \pi\mu.C.\text{cosec}\ \alpha.R^2$$

Substituting the value of *C*, we have

$$T = \pi\mu \times \frac{W}{2\pi R} \times \text{cosec } \alpha . R^2 = \frac{1}{2} \times \mu . W . R \text{ cosec } \alpha = \frac{1}{2} \times \mu . W . l$$

Trapezoidal or Truncated Conical Pivot Bearing

If the pivot bearing is not conical, but a frustum of a cone with *r*1 and *r*2, the external and internal radius respectively as shown in Fig, then



Area of the bearing surface,

$$A = \pi[(r_1)^2 - (r_2)^2]$$

$\therefore$ Intensity of uniform pressure,

$$p_n = \frac{W}{A} = \frac{W}{\pi[(r_1)^2 - (r_2)^2]} \qquad \ldots(i)$$

1. **Considering uniform pressure**

The total torque acting on the bearing is obtained by integrating the value of T_r, within the limits r_1 and r_2.

Total torque acting on the bearing,

$$T = \int_{r_2}^{r_1} 2\pi\mu . p_n \text{ cosec } \alpha . r^2 . dr = 2\pi\mu . p_n . \text{cosec } \alpha \left[\frac{r^3}{3}\right]_{r_2}^{r_1}$$

$$= 2\pi\mu . p_n . \text{cosec } \alpha \left[\frac{(r_1)^3 - (r_2)^3}{3}\right]$$

Substituting the value of p_n from equation ***(i)*,**

$$T = 2\pi\mu \times \frac{W}{\pi[(r_1)^2 - (r_2)^2]} \times \text{cosec } \alpha \left[\frac{(r_1)^3 - (r_2)^3}{3}\right]$$

$$= \frac{2}{3} \times \mu . W . \text{cosec } \alpha \left[\frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2}\right]$$

2. Considering uniform wear

the load transmitted to the ring,

$\delta W = 2\pi C.dr$

Total load transmitted to the ring,

$$W = \int_{r_2}^{r_1} 2\pi\ C.dr = 2\pi C[r]_{r_2}^{r_1} = 2\pi C(r_1 - r_2)$$

$$C = \frac{W}{2\pi(r_1 - r_2)}$$

We know that the torque acting on the ring, considering uniform wear, is

$$T_r = 2\pi\ \mu.C \text{ cosec } \alpha.r.dr$$

Total torque acting on the bearing,

$$T = \int_{r_2}^{r_1} 2\pi\ \mu.C \text{ cosec } \alpha.r.dr = 2\pi\ \mu.C.\text{cosec } \alpha \left[\frac{r^2}{2}\right]_{r_2}^{r_1}$$

$$= \pi\ \mu.C.\text{cosec } \alpha\left[(r_1)^2 - (r_2)^2\right]$$

Substituting the value of C from equation ***(ii)*,** we get

$$T = \pi\mu \times \frac{W}{2\pi(r_1 - r_2)} \times \text{cosec}\,\alpha\ [(r_1)^2 - (r_2)^2]$$

$$= \frac{1}{2} \times \mu.W\ (r_1 + r_2)\ \text{cosec }\alpha = \mu.W.R \text{ cosec } \alpha$$

$$R = \text{Mean radius of the bearing } = \frac{r_1 + r_2}{2}$$

PROBLEMS

Example 1. A conical pivot supports a load of 20 kN, the cone angle is 120º and the intensity of normal pressure is not to exceed 0.3 N/mm2. The external diameter is twice the internal diameter. Find the outer and inner radii of the bearing surface. If the shaft rotates at 200 r.p.m. and the coefficient of friction is 0.1, find the power absorbed in friction. Assume uniform pressure.

Solution. Given : W = 20 kN = 20 × 103 N ; 2 α= 120º or α= 60º ; p_n= 0.3 N/mm^2 ;N = 200 r.p.m. or ω= 2π× 200/60 = 20.95 rad/s ; μ = 0.1

Outer and inner radii of the bearing surface.

Let r_1 and r_2 = Outer and inner radii of the bearing surface, in mm.
Since the external diameter is twice the internal diameter, therefore

$r_1 = 2\,r_2$

We know that intensity of normal pressure (p_n),

$$0.3 = \frac{W}{\pi[(r_1)^2 - (r_2)^2]} = \frac{20 \times 10^3}{\pi[(2r_2)^2 - (r_2)^2]} = \frac{2.12 \times 10^3}{(r_2)^2}$$

$$(r_2)^2 = 2.12 \times 10^3 / 0.3 = 7.07 \times 10^3 \quad \text{or} \quad r_2 = 84 \text{ mm} \quad \textbf{Ans.}$$

$$r_1 = 2\,r_2 = 2 \times 84 = 168 \text{ mm} \quad \textbf{Ans.}$$

Power absorbed in friction

We know that total frictional torque (assuming uniform pressure),

$$T = \frac{2}{3} \times \mu.W.\text{cosec}\,\alpha \left[\frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2}\right]$$

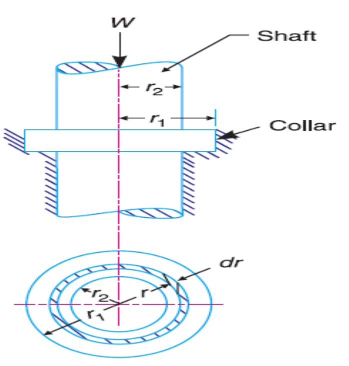
$$= \frac{2}{3} \times 0.1 \times 20 \times 10^3 \times \text{cosec}\,60^\circ = \left[\frac{(168)^3 - (84)^3}{(168)^2 - (84)^2}\right] \text{N-mm}$$

$$= 301760 \text{ N-mm} = 301.76 \text{ N-m}$$

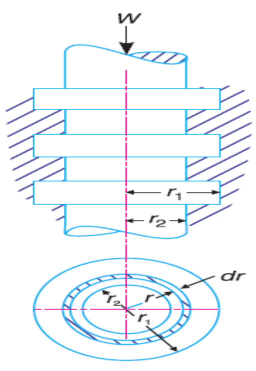
Power absorbed in friction
$P = T.\omega = 301.76 \times 20.95 = 6322$ W = 6.322 kW

Flat Collar Bearing

We have already discussed that collar bearings are used to take the axial thrust of the rotating hafts. There may be a single collar or multiple collar bearings as shown in Fig.(*a*) and (*b*)respectively. The collar bearings are also known as ***thrust bearings.*** The friction in the collar bearings may be found as discussed below :



(*a*) Single collar bearing



(*b*) Multiple collar bearing.

Consider a single flat collar bearing supporting a shaft as shown in Fig(*a*).

Let r_1 = External radius of the collar,
r_2 = Internal radius of the collar.
Area of the bearing surface,

$$A = \pi [(r_1)^2 - (r_2)^2]$$

1. **Considering uniform pressure**

When the pressure is uniformly distributed over the bearing surface, then the intensity of pressure,

$$p = \frac{W}{A} = \frac{W}{\pi[(r_1)^2 - (r_2)^2]} \qquad \ldots(i)$$

the frictional torque on the ring of radius r and thickness dr,

$$T_r = 2\pi\mu.p.r^2.dr$$

Integrating this equation within the limits from r_2 to r_1 for the total frictional torque on the collar.

Total frictional torque,

$$T = \int_{r_2}^{r_1} 2\pi\mu.p.r^2.dr = 2\pi\mu.p\left[\frac{r^3}{3}\right]_{r_2}^{r_1} = 2\pi\mu.p\left[\frac{(r_1)^3 - (r_2)^3}{3}\right]$$

Substituting the value of p from equation **(*i*),**

$$T = 2\pi\mu \times \frac{W}{\pi[(r_1)^2 - (r_2)^2]}\left[\frac{(r_1)^3 - (r_2)^3}{3}\right]$$

$$= \frac{2}{3} \times \mu.W\left[\frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2}\right]$$

2. Considering uniform wear

the load transmitted on the ring, considering uniform wear is,

$$\delta W = p_r.2\pi r.dr = \frac{C}{r} \times 2\pi r.dr = 2\pi C.dr$$

Total load transmitted to the collar,

$$W = \int_{r_2}^{r_1} 2\pi C.dr = 2\pi C[r]_{r_2}^{r_1} = 2\pi C(r_1 - r_2)$$

$$C = \frac{W}{2\pi(r_1 - r_2)} \quad ...(ii)$$

We also know that frictional torque on the ring,

We also know that frictional torque on the ring,

$$T_r = \mu.\delta W.r = \mu \times 2\pi C.dr.r = 2\pi\mu.C.r.dr$$

Total frictional torque on the bearing,

$$T = \int_{r_2}^{r_1} 2\pi\mu C.r.dr = 2\pi\mu.C\left[\frac{r^2}{2}\right]_{r_2}^{r_1} = 2\pi\mu.C\left[\frac{(r_1)^2 - (r_2)^2}{2}\right]$$

$$= \pi\mu.C[(r_1)^2 - (r_2)^2]$$

Substituting the value of C from equation ***(ii)*,**

$$T = \pi\mu \times \frac{W}{2\pi(r_1 - r_2)}[(r_1)^2 - (r_2)^2] = \frac{1}{2} \times \mu.W(r_1 + r_2)$$

PROBLEMS

Example 1. A thrust shaft of a ship has 6collars of 600 mm external diameter and 300 mm internal diameter. The total thrust from the propeller is 100 kN. If the coefficient of friction is 0.12 and speed of the engine90 r.p.m., find the power absorbed in friction at the thrust block, assuming

l. uniform pressure

2. uniform wear.

Solution. Given : $n = 6$; $d_1 = 600$ mm or $r_1 = 300$ mm ; $d_2 = 300$ mm or $r_2 = 150$ mm ; $W = 100$ kN= 100×10^3 N ; $\mu = 0.12$; $N = 90$ r.p.m. or $\omega = 2\pi \times 90/60 = 9.426$ rad/s

1. **Power absorbed in friction, assuming uniform pressure**

We know that total frictional torque transmitted,

$$T = \frac{2}{3} \times \mu.W\left[\frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2}\right]$$

$$= \frac{2}{3} \times 0.12 \times 100 \times 10^3\left[\frac{(300)^3 - (150)^3}{(300)^2 - (150)^2}\right] = 2800 \times 10^3 \text{ N-mm}$$

$$= 2800 \text{ N-m}$$

Power absorbed in friction,

$P = T\omega = 2800 \times 9.426 = 26\ 400$ W = 26.4 kW

2. Power absorbed in friction assuming uniform wear

We know that total frictional torque transmitted,

$$T = \frac{1}{2} \times \mu.W\,(r_1 + r_2) = \frac{1}{2} \times 0.12 \times 100 \times 10^3\ (300 + 150)\ \text{N-mm}$$
$$= 2700 \times 10^3\ \text{N-mm} = 2700\ \text{N-m}$$

Power absorbed in friction,

$P = T.\omega = 2700 \times 9.426 = 25\ 450$ W = 25.45 kW

Example 2. A shaft has a number of a collars integral with it. The external diameter of the collars is 400 mm and the shaft diameter is 250 mm. If the intensity of pressure is 0.35 N/mm^2(uniform) and the coefficient of friction is 0.05, estimate :
1. power absorbed when the shaft runs at105 r.p.m. carrying a load of 150 kN ;
2. number of collars required.

Solution. Given : d_1 = 400 mm or r_1 = 200 mm ; d_2 = 250 mm or r_2 = 125 mm ; p = 0.35N/mm^2 ; μ = 0.05 ; N = 105 r.p.m or
$\omega = 2\pi \times 105/60 = 11$ rad/s ; W = 150 kN = 150×10^3 N

1. Power absorbed
We know that for uniform pressure, total frictional torque transmitted

$$T = \frac{2}{3} \times \mu.W \left[\frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2}\right] = \frac{2}{3} \times 0.05 \times 150 \times 10^3 \left[\frac{(200)^3 - (125)^3}{(200)^2 - (125)^2}\right] \text{N-mm}$$
$$= 5000 \times 248 = 1240 \times 10^3\ \text{N-mm} = 1240\ \text{N-m}$$

Power absorbed,

$P = T\omega = 1240 \times 11 = 13640$ W = 13.64 kW

2.**Number of collars required**

Let n = Number of collars required.
We know that the intensity of uniform pressure (p),

$$0.35 = \frac{W}{n.\pi[(r_1)^2 - (r_2)^2]} = \frac{150 \times 10^3}{n.\pi[(200)^2 - (125)^2]} = \frac{1.96}{n}$$
$$n = 1.96/0.35 = 5.6 \text{ say } 6 \textbf{ Ans.}$$

UNIT –III

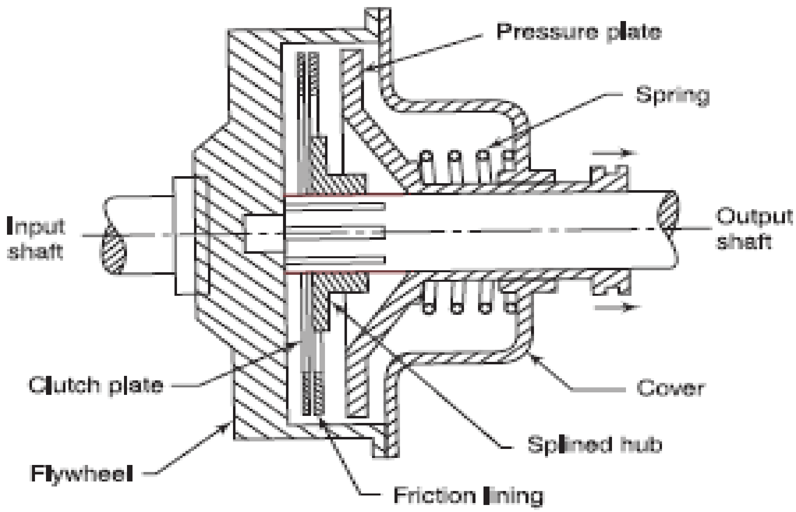
CLUTCHES

FRICTION CLUTCHES

A clutch is a device used to transmit the rotary motion of one shaft to another when desired. The axes of the two shafts are coincident. In friction clutches, the connection of the engine shaft to the gear box shaft is affected by friction between two or more rotating concentric surfaces. The surfaces can be pressed firmly against one another when engaged and the clutch tends to rotate as a single unit.

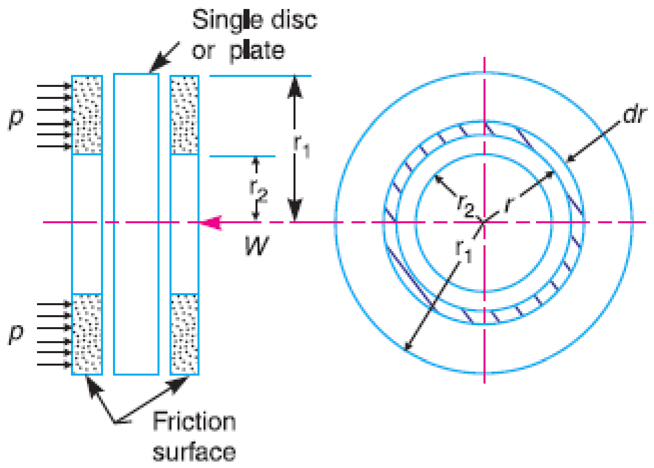
SINGLE PLATE CLUTCH (DISC CLUTCH)

A disc clutch consists of a clutch plate attached to a splined hub which is free to slide axially on splines cut on the driven shaft. The clutch plate is made of steel and has a ring of friction lining on each side. The engine shaft supports a rigidly fixed flywheel. A spring-loaded pressure plate presses the clutch plate firmly against the flywheel when the clutch is engaged. When disengaged, the springs press against a cover attached to the flywheel. Thus, both the flywheel and the pressure plate rotate with the input shaft. The movement of the clutch pedal is transferred to the pressure plate through a thrust bearing. Figure 8.13 shows the pressure plate pulled back by the release levers and the friction linings on the clutch plate are no longer in contact with the pressure plate or the flywheel. The flywheel rotates without driving the clutch plate and thus, the driven shaft.



When the foot is taken off the clutch pedal, the pressure on the thrust bearing is released. As a result, the springs become free to move the pressure plate to bring it in contact with the clutch plate. The clutch plate slides on the splined hub and is tightly gripped between the pressure plate and the fly wheel. The friction between the linings on the clutch plate, and the flywheel on one side and the pressure plate on the other, cause the clutch plate and hence, the driven shaft to rotate. In case the resisting torque on the driven shaft exceeds the torque at the clutch, clutch slip will occur.

Torque transmitted by plate or disc clutch



The following notations are used in the derivation

T= Torque transmitted by the clutch

P= intensity of axial pressure

r1&r2=external and internal radii of friction faces

μ= co-efficient of friction

Consider an elemental ring of radius r and thickness dr

Friction surface = 2πrdr

Axial force on the dw= pressure *area

= P*2πrdr

Frictional force acting on the ring tangentially at radius r

Fr= μdw=μ*p*2πrdr

Frictional torque acting on the ring $T_r=F_r*r=\mu p*2\pi r*dr*r=2\pi\mu pr^2dr$

Considering uniform pressure

When the pressure is uniformly distributed over the entire area of the friction face, then the intensity of pressure,

$$P=W/\pi[(r_1)^2-(r_2)^2]\text{-------------(i)}$$

Where W = Axial thrust with which the contact or friction surfaces are held together.
We have discussed above that the frictional torque on the elementary ring of radius r and thickness dr is

$T_r = 2\,\pi\mu.p.r^2\,dr$

Integrating this equation within the limits from r_2 to r_1 for the total frictional torque.

Therefore Total frictional torque acting on the friction surface or on the clutch,

$$T = \int_{r_1}^{r_2} 2\pi\mu.p.r^2.dr = 2\,\pi\mu\,p\left[\frac{r_3}{3}\right]_{r_2}^{r_1} = 2\pi\mu\,p\left[\frac{(r_1)^3 - (r_2)^3}{3}\right]$$

Substituting the value of p from equation ***(i)*****,**

$$T = 2\pi\mu \times \frac{W}{\pi[(r_1)^2 - (r_2)^2]} \times \frac{(r_1)^3 - (r_2)^3}{3}$$

$$= \frac{2}{3} \times \mu.W\left[\frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2}\right] = \mu.W.R$$

R = Mean radius of friction surface

$$= \frac{2}{3}\left[\frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2}\right]$$

2. ***Considering uniform wear***

Let p be the normal intensity of pressure at a distance r from the axis of the Clutch. Since the intensity of pressure varies inversely with the distance, therefore

$p.r. = C$ (a constant) or $p = C/r$

and the normal force on the ring,

$$\delta W = p.2\pi r.dr = \frac{C}{r} \times 2\pi C.dr = 2\pi C.dr$$

∴ Total force acting on the friction surface,

$$W = \int_{r_2}^{r_1} 2\pi C\,dr = 2\pi C[r]_{r_2}^{r_1} = 2\pi C(r_1 - r_2)$$

$$C = \frac{W}{2\pi(r_1 - r_2)}$$

We know that the frictional torque acting on the ring,

$$T_r = 2\pi\mu.p\,r^2.dr = 2\pi\mu \times \frac{C}{r} \times r^2.dr = 2\pi\mu.C.r.dr$$

Total frictional torque on the friction surface,

$$T = \int_{r_2}^{r_1} 2\pi\mu.C.r.dr = 2\pi\mu.C\left[\frac{r^2}{2}\right]_{r_2}^{r_1} = 2\pi\mu.C\left[\frac{(r_1)^2 - (r_2)^2}{2}\right]$$

$$= \pi\mu.C[(r_1)^2 - (r_2)^2] = \pi\mu \times \frac{W}{2\pi(r_1 - r_2)}[(r_1)^2 - (r_2)^2]$$

$$= \frac{1}{2} \times \mu.W(r_1 + r_2) = \mu.W.R$$

R = Mean radius of the friction surface = $(r_1 + r_2)/2$

Multiple plate clutch

In a multi-plate clutch, the number of frictional linings and the metal plates is Increased which increases the capacity of the clutch to transmit torque. Figure8.14 show a simplified diagram of a multi-plate clutch. The friction rings are splined on their outer circumference and engage with corresponding splines on the flywheel. They are free to slide axially.

The Friction material thus, rotates with the flywheel and the engine shaft. The Number of friction rings depends upon the torque to be transmitted.

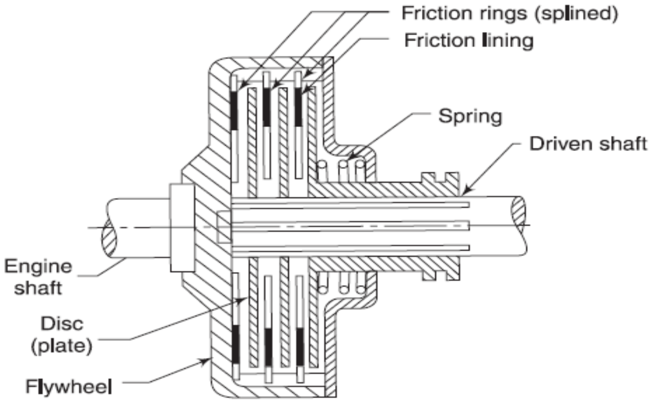


Fig. 8.14

The driven shaft also supports discs on the splines which rotate with the driven shaft and can slide axially. If the actuating force on the pedal is removed, a spring presses the discs into contact with the friction rings and the torque is transmitted between the engine shaft and the driven shaft. If n is the total number of plates both on the driving and the driven members, the number of active surfaces will be $n - 1$.

Let n_1 = Number of discs on the driving shaft, and
n_2 = Number of discs on the driven shaft.

Number of pairs of contact surfaces,

$$n = n_1 + n_2 - 1$$

And total frictional torque acting on the friction surfaces or on the clutch,

$$T = n.\mu.W.R$$

Where R = Mean radius of the friction surfaces

$$= \frac{2}{3}\left[\frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2}\right]$$

$$= \frac{r_1 + r_2}{2}$$

PROBLEMS

Example1. Determine the maximum, minimum and average pressure in plate clutch when the axial force is 4 kN. The inside radius of the contact surface is 50 mm and the outside radius is 100 mm. Assume uniform wear.

Solution. Given: W = 4 kN = 4 × 103 N, r_2 = 50 mm ; r_1 = 100 mm

Maximum pressure

Let p_{max} = Maximum pressure.

Since the intensity of pressure is maximum at the inner radius (r_2), therefore

$$p_{max} \times r2 = C \text{ or } C = 50\, p_{max}$$

We know that the total force on the contact surface (W),

$4 \times 103 = 2\,\pi\,C\,(r_1 - r_2) = 2\,\pi \times 50\,p_{max}\,(100 - 50) = 15\,710\,p_{max}$

$P_{max} = 4 \times 103/15\,710 = 0.2546$ N/mm^2

Minimum pressure

Let p_{min} = Minimum pressure.

Since the intensity of pressure is minimum at the outer radius (r_1),

Therefore $P_{min} \times r_1 = C$ or $C = 100\,p_{min}$

We know that the total force on the contact surface (W),

$4 \times 103 = 2\,\pi\,C\,(r_1 - r_2) = 2\pi \times 100\,p_{min}\,(100 - 50) = 31\,420\,p_{min}$

$P_{min} = 4 \times 103/31\,420 = 0.1273$ N/mm^2

Average pressure

We know that average pressure,

$$p_{av} = \frac{\text{Total normal force on contact surface}}{\text{Cross-sectional area of contact surfaces}}$$

$$= \frac{W}{\pi[(r_1)^2 - (r_2)^2]} = \frac{4 \times 10^3}{\pi[(100)^2 - (50)^2]} = 0.17 \text{ N/mm}^2$$

Example2. A single plate clutch, with both sides effective, has outer and inner diameters 300 mm and 200 mm respectively. The maximum intensity of pressure at any point in the contact surface is not to exceed 0.1 N/mm2. If the coefficient of friction is 0.3, determine the power transmitted by a clutch at a speed 2500 r.p.m.

Solution. Given: d_1 = 300 mm or r_1 = 150 mm ; d_2 = 200 mm or r_2 = 100 mm ;
p = 0.1 N/mm^2; μ = 0.3 ; N= 2500 r.p.m. or ω= 2π× 2500/60 = 261.8 rad/s
Since the intensity of pressure (p) is maximum at the inner radius (r_2), therefore for uniform

$p.r_2 = C$ or $C = 0.1 \times 100 = 10$ N/mm
We know that the axial thrust,
$W = 2\,\pi\, C\,(r_1 - r_2) = 2\,\pi \times 10\,(150 - 100) = 3142$ N
Mean radius of the friction surfaces for uniform wear,

$$R = \frac{r_1 + r_2}{2} = \frac{150 + 100}{2} = 125 \text{ mm} = 0.125\,\text{m}$$

We know that torque transmitted,
$T = n.\mu.W.R = 2 \times 0.3 \times 3142 \times 0.125 = 235.65$ N-m
Power transmitted by a clutch,
$P = T*\omega = 235.65 \times 261.8 = 61\,693$ W = 61.693 kW

CONE CLUTCH

A cone clutch, as shown in Fig. 10.24, was extensively used in automobiles but now-a-days it has been replaced completely by the disc clutch

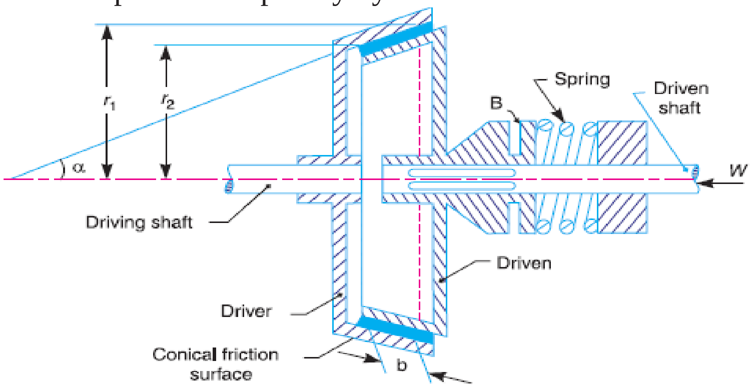


Fig. 10.24. Cone clutch.

It consists of one pair of friction surface only. In a cone clutch, the driver is keyed to the driving shaft by a sunk key and has an inside conical surface or face which exactly fits into the outside conical surface of the driven.

The driven member resting on the feather key in the driven shaft, maybe shifted along the shaft by a forked lever provided at *B*, in order to engage the clutch by bringing the two conical surfaces in contact. Due to the frictional resistance set up at this contact surface, the torque is transmitted from one shaft to another. In some cases, a spring is placed around the driven shaft in contact with the hub of the driven. This spring holds the clutch faces in contact and maintains the pressure between them, and the forked lever is used only for disengagement of the clutch. The contact surfaces of the clutch may be metal to metal contact, but more often the driven member islined with some material like wood, leather, cork or asbestos etc. The material of the clutch faces (*i.e* .contact surfaces) depends upon the allowable normal pressure and the coefficient of friction.

Consider a pair of friction surface as shown in Fig. Since the area of contact of a pair of friction surface is a frustum of a cone, therefore the torque transmitted by the cone clutch maybe determined in the similar manner as discussed.

Let p_n = Intensity of pressure with which the conical friction surfaces are held together (*i.e.* normal pressure between contact surfaces),

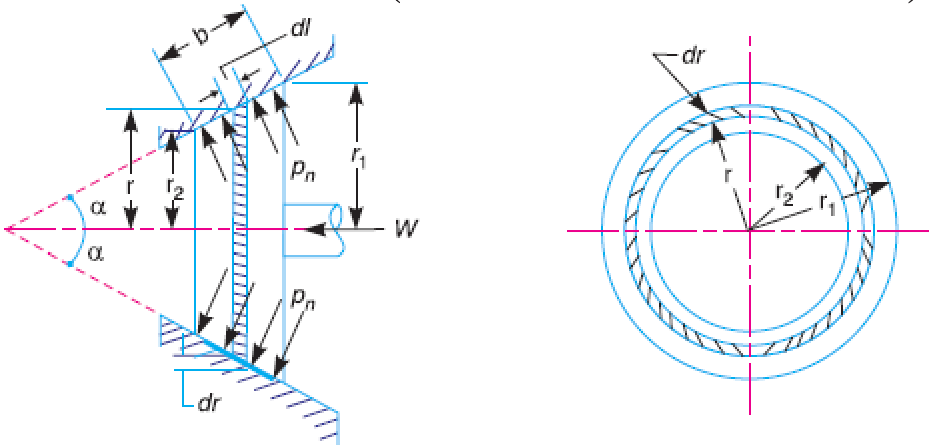
r_1 and r_2 = Outer and inner radius of friction surfaces respectively

R = Mean radius of the friction surface=$(r_1 + r_2)/2$

α= Semi angle of the cone (also called face angle of the cone) or the angle of the friction surface with the axis of the clutch,

μ = Coefficient of friction between contact surfaces, and

b = Width of the contact surfaces (also known as face width or clutch face).



(*a*) (*b*)

Consider a small ring of radius *r* and thickness *dr*, as shown in Fig. 10.25 (*b*).

Let *dl* is length of ring of the friction surface, such that

$$dl = dr.\text{cose } \alpha$$

Area of the ring= $A = 2\pi\, r.dl = 2\pi r.dr$ cosec α

We shall consider the following two cases :

1. When there is a uniform pressure, and

2. When there is a uniform wear.

Considering uniform pressure

We know that normal load acting on the ring,

δW_n = Normal pressure × Area of ring = $p_n \times 2\pi r.dr.\text{cosec}\,\alpha$

The axial load acting on the ring,

δW = Horizontal component of δW_n (*i.e.* in the direction of W)

$= \delta W_n \times \sin\alpha = p_n \times 2\pi r.dr.\,\text{cosec}\,\alpha \times \sin\alpha = 2\pi \times p_n.r.dr$

Total axial load transmitted to the clutch or the axial spring force required,

$$W = \int_{r_2}^{r_1} 2\pi p_n.r.dr = 2\pi p_n\left[\frac{r^2}{2}\right]_{r_2}^{r_1} = 2\pi p_n\left[\frac{(r_1)^2 - (r_2)^2}{2}\right]$$

$$= \pi p_n\left[(r_1)^2 - (r_2)^2\right]$$

$$p_n = \frac{W}{\pi[(r_1)^2 - (r_2)^2]}$$

We know that frictional force on the ring acting tangentially at radius r,

$F_r = \mu.\delta W_n = \mu.p_n \times 2\pi r.dr.\text{cosec}\,\alpha$

Frictional torque acting on the ring,

$T_r = F_r \times r = \mu.p_n \times 2\pi r.dr.\,\text{cosec}\,\alpha.r = 2\pi\mu.p_n.\text{cosec}\,\alpha.r^2\,dr$

Integrating this expression within the limits from r_2 to r_1 for the total frictional torque on the clutch.

∴ Total frictional torque,

$$T = \int_{r_2}^{r_1} 2\pi\mu.p_n.\text{cosec}\,\alpha.r^2.dr = 2\pi\mu\, p_n.\text{cosec}\,\alpha\left[\frac{r^3}{3}\right]_{r_2}^{r_1}$$

$$= 2\pi\mu\; p_n.\text{cosec}\,\alpha\left[\frac{(r_1)^3 - (r_2)^3}{3}\right]$$

Substituting the value of p_n from equation (*i*), we get

$$T = 2\pi\mu \times \frac{W}{\pi[(r_1)^2 - (r_2)^2]} \times \text{cosec}\,\alpha\left[\frac{(r_1)^3 - (r_2)^3}{3}\right]$$

$$= \frac{2}{3} \times \mu.W.\text{cosec}\,\alpha\left[\frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2}\right]$$

Considering uniform wear

In Fig. 10.25, let pr be the normal intensity of pressure at a distance r from the axis of the clutch. We know that, in case of uniform wear, the intensity of pressure varies inversely with the distance.

$P_r.r = C$ (a constant) or $p_r = C / r$

We know that the normal load acting on the ring,

δW_n = Normal pressure × Area of ring = $pr \times 2\pi r.dr\,\text{cosec}\alpha$

The axial load acting on the ring ,

$\delta W = \delta W_n \times \sin\alpha = p_r.2\pi r.dr.\text{cosec}\,\alpha.\sin\alpha = p_r \times 2\pi r.dr$

$$= \frac{C}{r} \times 2\pi r.dr = 2\pi C.dr$$

$\therefore$ Total axial load transmitted to the clutch,

$$W = \int_{r_2}^{r_1} 2\pi C.dr = 2\pi C[r]_{r_2}^{r_1} = 2\pi C(r_1 - r_2)$$

$$C = \frac{W}{2\pi(r_1 - r_2)}$$

We know that frictional force acting on the ring,

$F_r = \mu.\delta W_n = \mu.p_r \times 2\pi r \times dr$ cosecα

Frictional torque acting on the ring,

$T_r = F_r \times r = \mu.p_r \times 2\pi\, r.dr.$cosec$\alpha \times r$

$$= \mu \times \frac{C}{r} \times 2\pi r^2.dr.\text{cosec}\ \alpha = 2\pi\mu.C\ \text{cosec}\ \alpha \times r\ dr$$

$\therefore$ Total frictional torque acting on the clutch,

$$T = \int_{r_2}^{r_1} 2\pi\mu.C.\text{cosec}\ \alpha.r\,dr = 2\pi\mu.C.\text{cosec}\ \alpha\left[\frac{r^2}{2}\right]_{r_2}^{r_1}$$

$$= 2\pi\mu.C.\text{cosec}\ \alpha\left[\frac{(r_1)^2 - (r_2)^2}{2}\right]$$

Substituting the value of C from equation (*i*), we have

$$T = 2\pi\mu \times \frac{W}{2\pi(r_1 - r_2)} \times \text{cosec}\ \alpha\left[\frac{(r_1)^2 - (r_2)^2}{2}\right]$$

$$= \mu.W\ \text{cosec}\ \alpha\left(\frac{r_1 + r_2}{2}\right) = \mu.W.R\ \text{cosec}\ \alpha$$

$$R = \frac{r_1 + r_2}{2} = \text{Mean radius of friction surface}$$

PROBLEMS

Example 1. An engine developing 45 kW at 1000 r.p.m. is fitted with a cone clutch built inside the flywheel. The cone has a face angle of 12.5º and a maximum mean diameter of 500 mm. The coefficient of friction is 0.2. The normal pressure on the clutch face is not to exceed 0.1 N/mm 2. Determine: 1. the axial spring force necessary to engage to clutch, and 2. the face width required.

Solution. Given : P = 45 kW = 45 × 103 W ; N = 1000 r.p.m. or ω= 2π× 1000/60 = 104.7rad/s ; α= 12.5º ; D = 500 mm or R = 250 mm = 0.25 m ; μ = 0.2 ;
p_n=0.1N/mm^2

1. Axial spring force necessary to engage the clutch

First of all, let us find the torque (T) developed by the clutch and the normal load (W_n) acting on the friction surface.

We know that power developed by the clutch (P),

$45 \times 103 = T\omega = T \times 104.7$ or $T = 45 \times 10^3/104.7 = 430$ N-m

We also know that the torque developed by the clutch (T),

$430 = \mu.W_n .R = 0.2 \times W_n \times 0.25 = 0.05\ W_n$

$W_n = 430/0.05 = 8600$ N

Axial spring force necessary to engage the clutch,

$W_e = W_n (\sin \alpha + \mu \cos\alpha)$

$= 8600 (\sin 12.5^o + 0.2 \cos 12.5^o) = 3540$ N

2. ***Face width required***

Let b = Face width required

We know that normal load acting on the friction surface (W_n),

$8600 = p_n \times 2\pi R.b = 0.1 \times 2\pi \times 250 \times b = 157\ b$

$b = 8600/157 = 54.7$ mm

Example 2. A conical friction clutch is used to transmit 90 kW at 1500 r.p.m. The semi cone angle is 20º and the coefficient of friction is 0.2. If the mean diameter of the bearing surface is375 mm and the intensity of normal pressure is not to exceed 0.25 N/mm2, find the dimensions of the conical bearing surface and the axial load required.

Solution. Given: $P = 90$ kW $= 90 \times 10^3$ W ; $N = 1500$ r.p.m. or $\omega = 2\ \pi \times 1500/60 =$ 156rad/s ; $\alpha = 20^o$; $\mu = 0.2$; $D = 375$ mm or $R = 187.5$ mm ; $p_n = 0.25$ N/mm^2

Dimensions of the conical bearing surface

Let r_1 and r_2 = External and internal radii of the bearing surface respectively,

b = Width of the bearing surface in mm, and

T = Torque transmitted.

We know that power transmitted (P),

$90 \times 103 = T\omega = T \times 156$

$T = 90 \times 103/156 = 577$ N-m $= 577 \times 10^3$ N-mm

The torque transmitted (T),

$577 \times 103 = 2\ \pi\ \mu\ p_n.R^2.b = 2\pi \times 0.2 \times 0.25\ (187.5)^2\ b = 11\ 046\ b$

$b = 577 \times 10^3/11\ 046 = 52.2$ mm

We know that $r_1 + r_2 = 2R = 2 \times 187.5 = 375$ mm------------i

$r1 - r2 = b \sin \alpha = 52.2 \sin 20^o = 18$ mm------------------------ii

From equations **(*i*)** and **(*ii*),**

$r_1 = 196.5$ mm, and $r_2 = 178.5$ mm

Axial load required

Since in case of friction clutch, uniform wear is considered and the intensity of pressure is maximum at the minimum contact surface radius ($r2$), therefore

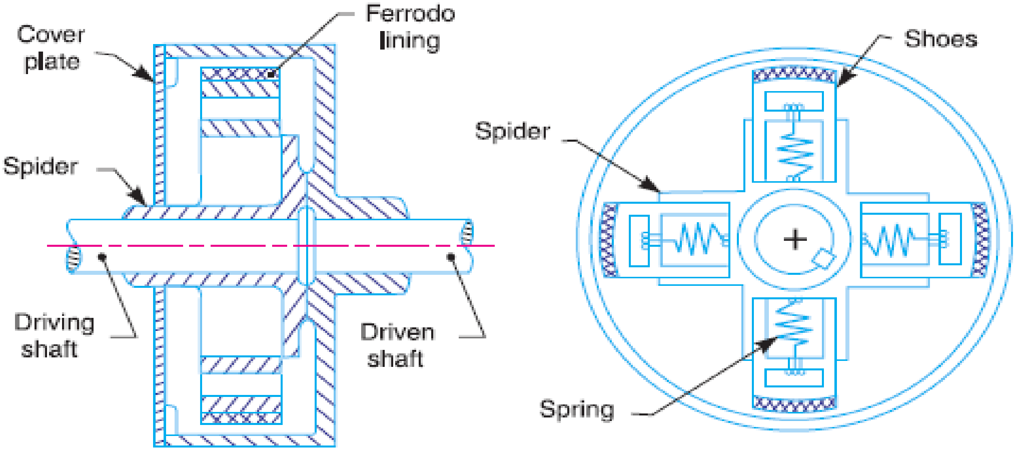
$p_n.r_2 = C$ (a constant) or $C = 0.25 \times 178.5 = 44.6$ N/mm

We know that the axial load required,

$W = 2\pi C (r_1 - r_2) = 2\pi \times 44.6 (196.5 - 178.5) = 5045$ N

Centrifugal Clutch

The centrifugal clutches are usually incorporated into the motor pulleys. It consists of a number of shoes on the inside of a rim of the pulley, as shown in Fig. 10.28. The outer surface of the shoes are covered with a friction material. These shoes, which can move radially in guides, are held



Against the boss (or spider) on the driving shaft by means of springs. The springs exert a radially inward force which is assumed constant. The mass of the shoe, when revolving, causes it to exert a radially outward force (*i.e.* centrifugal force). The magnitude of this centrifugal force depends upon the speed at which the shoe is revolving. A little consideration will show that when the centrifugal force is less than the spring force, the shoe remains in the same position as when the driving shaft was stationary, but when the centrifugal force is equal to the spring force, the shoe is just floating. When the centrifugal force exceeds the spring force, the shoe moves outward and comes into contact with the driven member and presses against it. The force with which the shoe presses against the driven member is the difference of the centrifugal force and the spring force. The increase of speed causes the shoe to press harder and enables more torque to be transmitted.

In order to determine the mass and size of the shoes, the following procedure is adopted:

Mass of the shoes

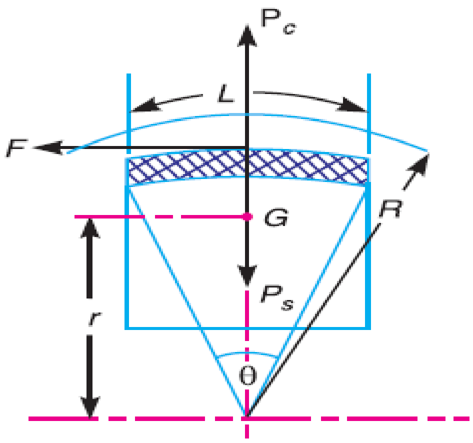


Fig. 10.29. Forces on a shoe of centrifugal clutch.

Consider one shoe of a centrifugal clutch as shown in Fig

Let m = Mass of each shoe,

n = Number of shoes,

r = Distance of centre of gravity of the shoe from the centre of the spider,

R = Inside radius of the pulley rim,

N = Running speed of the pulley in r.p.m.,

ω = Angular running speed of the pulley in rad/s

$= 2\pi N/60$ rad/s,

ω_1 = Angular speed at which the engagement begins to take place, and

$\propto$ = Coefficient of friction between the shoe and rim.

We know that the centrifugal force acting on each shoe at the running speed,

$$*P_c = m.\omega^2.r$$

and the inward force on each shoe exerted by the spring at the speed at which engagement begins to take place,

$$P_s = m(\omega_1)^2 r$$

∴ The net outward radial force (*i.e.* centrifugal force) with which

The shoe presses against the rim at the running speed

$$= P_c - P_s$$

The frictional force acting tangentially on each shoe,

$$F = \propto (P_c - P_s)$$

∴ Frictional torque acting on each shoe,

$$= F \times R = \propto (P_c - P_s) R$$

Total frictional torque transmitted,

$$T = \propto (P_c - P_s) R \times n = n.F.R$$

From this expression, the mass of the shoes (m) may be

evaluated.

Size of the shoes

Let l = Contact length of the shoes,

b = Width of the shoes,

R = Contact radius of the shoes. It is same as the inside radius of the rim of the pulley.

θ = Angle subtended by the shoes at the centre of the spider in radians.

p = Intensity of pressure exerted on the shoe. In order to ensure reason-able life, the intensity of pressure may be taken as 0.1 N/mm^2.

We know that $\theta = l/R$ rad or $l = \theta.R$

∴ Area of contact of the shoe,

$$A = l.b$$

The force with which the shoe presses against the rim

$$A \times p = l.b.p$$

Since the force with which the shoe presses against the rim at the running speed is $(P_c - P_s)$, therefore

$$l.b.p = P_c - P_s$$

PROBLEMS

Example 1. A centrifugal clutch is to transmit 15 kW at 900 r.p.m. The shoes are four in number. The speed at which the engagement begins is 3/4th of the running speed. The inside radius of the pulley rim is 150 mm and the centre of gravity of the shoe lies at 120 mm from the centre of the spider. The shoes are lined with Ferrodo for which the coefficient of friction may be taken as 0.25. Determine: **1.** Mass of the shoes, and **2.** Size of the shoes, if angle subtended by the shoes at the centre of the spider is 60º and the pressure exerted on the shoes is 0.1 N/mm^2.

Solution. Given : P = 15 kW = 15 × 10^3 W ; N = 900 r.p.m. or ω = 25 × 900/60 = 94.26 rad/s ; n = 4 ; R = 150 mm = 0.15 m ; r = 120 mm = 0.12 m ; ∝ = 0.25

Since the speed at which the engagement begins (*i.e.* ω_1) is 3/4th of the running speed (*i.e.* ω), therefore

$$\omega_1 = \frac{3}{4}\,\omega = \frac{3}{4} \times 94.26 = 70.7 \text{ rad/s}$$

Let T = Torque transmitted at the running speed.

We know that power transmitted (P),

$= T.\omega = T \times 94.26$ or $T = 15 \times 10^3/94.26 =$
15 × 10 =159 N-m

1. ***Mass of the shoes***

Let m = Mass of the shoes in kg.

We know that the centrifugal force acting on each shoe,

$$P_c = m.\omega^2.r = m\,(94.26)^2 \times 0.12 = 1066\,m \text{ N}$$

the inward force on each shoe exerted by the spring *i.e*. the centrifugal force at the engagement speed ω_1,

$$P_s = m\,(\omega_1)^2\,r = m\,(70.7)^2 \times 0.12 = 600\,m \text{ N}$$

$\therefore$ Frictional force acting tangentially on each shoe,

$$F = \propto (P_c - P_s) = 0.25\,(1066\,m - 600\,m)\ 116.5\,m \text{ N}$$

We know that the torque transmitted (T),

$$159 = n.F.R = 4 \times 116.5\,m \times 0.15 = 70\,m \text{ or } m = 2.27 \text{ kg}$$

2. ***Size of the shoes***

Let l = Contact length of shoes in mm,
b = Width of the shoes in mm,

θ Angle subtended by the shoes at the centre of the spider in radians = 60º = $\pi/3$ rad, and

p = Pressure exerted on the shoes in N/mm^2 = 0.1 N/mm^2

We know that $$l = \theta.\,R = \frac{\pi}{3} \times 150 = 157.1 \text{ mm}$$

$$l.b.p = P_c - P_s = 1066\,m - 600\,m = 466\,m$$

$$\therefore \quad 157.1 \times b \times 0.1 = 466 \times 2.27 = 1058$$

$$b = 1058/157.1 \times 0.1 = 67.3 \text{ mm}$$

Example2. A centrifugal clutch has four shoes which slide radially in a spider keyed to the driving shaft and make contact with the internal cylindrical surface of a rim keyed to the driven shaft. When the clutch is at rest, each shoe is pulled against a stop by a spring so as to leave a radial clearance of 5 mm between the shoe and the rim. The pull exerted by the spring is then 500 N. The mass centre of the shoe is 160 mm from the axis of the clutch.

If the internal diameter of the rim is 400 mm, the mass of each shoe is 8 kg, the stiffness of each spring is 50 N/mm and the coefficient of friction between the shoe and the rim is 0.3 ; find the power transmitted by the clutch at 500 r.p.m.

Solution. Given : n = 4 ; c = 5 mm ; S = 500 N ; r = 160 mm ; D = 400 mm or R = 200 mm=0.2 m ; m = 8 kg ; s = 50 N/mm ; $\propto$ = 0.3 ; N = 500 r.p.m. or $\omega = 2\pi \times 500/60 = 52.37$ rad/s

We know that the operating radius,

$r_1 = r + c = 160 + 5 = 165$ mm = 0.165 m

Centrifugal force on each shoe,

$P_c = m\,.\omega^2.r_1 = 8\,(52.37)^2 \times 0.165 = 3620$ N

The inward force exerted by the spring,

$P_4 = S + c.s = 500 + 5 \times 50 = 750$ N

∴ Frictional force acting tangentially on each shoe,

$F = \propto (P_c - P_s) = 0.3\,(3620 - 750) = 861$ N

We know that total frictional torque transmitted by the clutch,

$T = n.F.R = 4 \times 861 \times 0.2 = 688.8$ N-m

∴ Power transmitted,

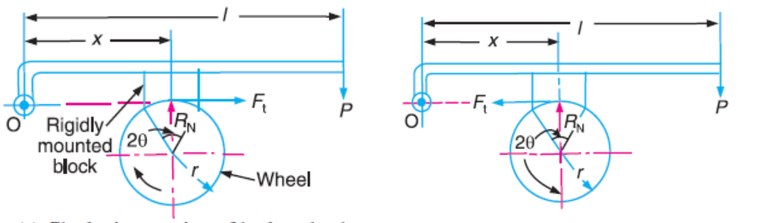
$P = T.\omega = 688.8 \times 52.37 = 36\,100$ W = 36.1 kW

BRAKES AND DYNAMOMETERS

A ***brake*** is a device by means of which artificial frictional resistance is applied to a moving machine member, in order to retard or stop the motion of a machine. In the process of performing this function, the brake absorbs either kinetic energy of the moving member or potential energy given up by objects being lowered by hoists, elevators etc

Single Block or Shoe Brake

A single block or shoe brake is shown in Fig. It consists of a block or shoe which is pressed against the rim of a revolving brake wheel drum. The block is made of a softer material than the rim of the wheel. This type of a brake is commonly used on railway trains and tram cars. The friction between the block and the wheel causes a tangential braking force to act on the wheel, which retard the rotation of the wheel. The block is pressed against the wheel by a force applied to one end of a lever to which the block is rigidly fixed as shown in Fig. 19.1. The other end of the lever is pivoted on a fixed fulcrum *O*.



(*a*) Clockwise rotation of brake wheel (*b*) Anticlockwise rotation of brake wheel.

Let P = Force applied at the end of the lever

R_N = Normal force pressing the brake block on the wheel,

r = Radius of the wheel,

2θ = Angle of contact surface of the block, μ = Coefficient of friction, and

F_t = Tangential braking force or the frictional force acting at the contact surface of the block and the wheel

If the angle of contact is less than 60°, then it may be assumed that the normal pressure between the block and the wheel is uniform. In such cases, tangential braking force on the wheel,

$$F_t = \mu.R_N \qquad ...(i)$$

The braking torque, $T_B = F_t.r = \mu.R_N.r \qquad ... (ii)$

Let us now consider the following three cases:

Case1. When the line of action of tangential braking force (Ft) passes through the fulcrum O of the lever, and the brake wheel rotates clockwise as shown in Fig. 19.1(a), then for equilibrium, taking moments about the fulcrum O, we have

$$R_N \times x = P \times l \text{ or } R_N = \frac{P \times l}{x}$$

Braking torque,

$$T_B = \mu.R_N.r = \mu \times \frac{P.l}{x} \times r = \frac{\mu.P.l.r}{x}$$

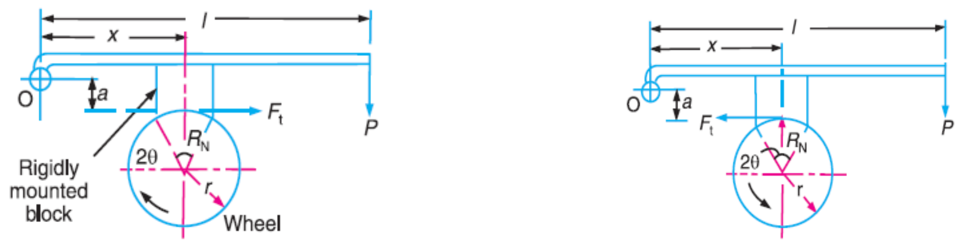
It may be noted that when the brake wheel rotates anticlockwise as shown in Fig. 19.1 (b), then the braking torque is same, *i.e.*

$$T_B = \mu.R_N.r = \frac{\mu.P.l.r}{x}$$

Case2. When the line of action of the tangential braking force (Ft) passes through a distance 'a' below the fulcrum O, and the brake wheel rotates clockwise as shown in Fig. 19.2 (a), then for equilibrium, taking moments about the fulcrum O,

$$R_N \times x + F_t \times a = P.l \quad \text{or} \quad R_N \times x + \mu\, R_N \times a = P.l \quad \text{or} \quad R_N = \frac{P.l}{x + \mu.a}$$

$$T_B = \mu R_N.r = \frac{\mu.\, p.l.r}{x + \mu.a}$$



(a) Clockwise rotation of brake wheel. (b) Anticlockwise rotation of brake wheel.

When the brake wheel rotates anticlockwise, as shown in Fig. 19.2 (b), then for equilibrium,

$$R_N.x = P.l + F_t.a = P.l + \mu.R_N.a$$

$$R_N (x - \mu.a) = P.l \quad \text{or} \quad R_N = \frac{P.l}{x - \mu.a}$$

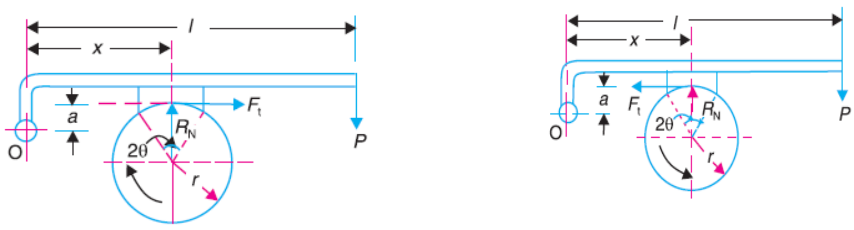
$$T_B = \mu.R_N.r = \frac{\mu.P.l.r}{x - \mu.a}$$

Case 3. When the lie of action of the tangential braking force (Ft) passes through a distance 'a' above the fulcrum O, and the brake wheel rotates clockwise as shown in Fig. 19.3 (a), then fore equilibrium, taking moments about the fulcrum O, we have

$$R_N.x = P.l + Ft.a = P.l + \mu.R_N.a$$

$$R_N (x - \mu.a) = P.l$$

$$R_N = \frac{P.l}{x - \mu.a}$$



(a) Clockwise rotation of brake wheel.　　(b) Anticlockwise rotation of brake wheel.

$$T_B = \mu.R_N.r = \frac{\mu.P.l.r}{x - \mu.a}$$

When the brake wheel rotates anticlockwise as shown in Fig. 19.3 (b), then for equilibrium, taking moments about the fulcrum O, we have

$$R_N \times x + Ft \times a = P.l$$
$$R_N \times x + \mu.R_N \times a = P.l$$

$$R_N = \frac{P.l}{x + \mu.a} \qquad T_B = \mu.R_N.r = \frac{\mu.P.l.r}{x + \mu.a}$$

Pivoted Block or Shoe Brake

We have discussed in the previous article that when the angle of contact is less than 60°, then it may be assumed that the normal pressure between the block and the wheel is uniform. But when the angle of contact is greater than 60°, then the unit pressure normal to the surface of contact is less at the ends than at the centre. In such cases, the block or shoe is pivoted to the lever, as shown in Fig.

instead of being rigidly attached to the lever. This gives uniform wear of the brake lining in the direction of the applied force. The braking torque or a pivoted block

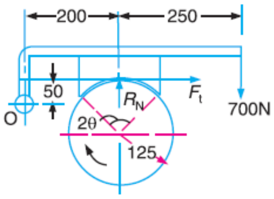
$$T_B = F_t \times r = \mu' . R_N . r$$

μ' = Equivalent coefficient of friction = $\dfrac{4\mu \sin\theta}{2\theta + \sin 2\theta}$,

μ = Actual coefficient of friction.

PROBLEMS

Example1. A single block brake is shown in Fig. 19.5.The diameter of the drum is 250 mm and the angle of contact is90°. If the operating force of 700 N is applied at the end of a lever and the coefficient of friction between the drum and the lining is0.35,Determine the torque that may be transmitted by the block brake.



All dimensions in mm.

Fig. 19.5

Solution. Given : d = 250 mm or r = 125 mm ; $2\theta = 90° = \pi / 2$ rad ; P = 700 N ; μ = 0.35

Since the angle of contact is greater than 60°, therefore equivalent coefficient of friction,

$$\mu' = \frac{4\mu \sin\theta}{2\theta + \sin 2\theta} = \frac{4 \times 0.35 \times \sin 45°}{\pi/2 + \sin 90°} = 0.385$$

R_N = Normal force pressing the block to the brake drum, and

F_t = Tangential braking force = $\mu'.R_N$

Taking moments about the fulcrum O, we have

$$700(250 + 200) + F_t \times 50 = R_N \times 200 = \frac{F_t}{\mu'} \times 200 = \frac{F_t}{0.385} \times 200 = 520\ F_t$$

$520\ F_t - 50F_t = 700 \times 450$ or $F_t = 700 \times 450/470 = 670$ N

We know that torque transmitted by the block brake,

$T_B = F_t \times r = 670 \times 125 = 8\ 3750$ N-mm = 83.75N-m

Example 2. A bicycle and rider of mass 100 kg are travelling at the rate of 16 km/h on a level road. A brake is applied to the rear wheel which is 0.9 m in diameter and this is the only resistance acting. How far will the bicycle travel and how many turns will it make before it comes to rest? The pressure applied on the brake is 100 N and μ = 0.05.

Solution. Given: m = 100 kg, v = 16 km / h = 4.44 m / s ; D = 0.9 m ; R N = 100 N; μ = 0.05

Distance travelled by the bicycle before it comes to rest

Let x = Distance travelled (in meters) by the bicycle before it comes to rest.

We know that tangential braking force acting at the point of contact of the brake wheel,

$$F_t = \mu . R_N = 0.05 \times 100 = 5 \text{ N}$$

$$= F_t \times x = 5 \times x = 5x \text{ N-m} \text{ ---------------(i)}$$

We know that kinetic energy of the bicycle

$$= \frac{m.v^2}{2} = \frac{100(4.44)^2}{2}$$

$$= 986 \text{ N-m} \quad \quad ...(ii)$$

In order to bring the bicycle to rest, the work done against friction must be equal to kinetic energy of the bicycle. Therefore equating equations **(*i*)** and **(*ii*),**

$$5x = 986 \text{ or } x = 986/5 = 197.2 \text{ m}$$

Number of revolutions made by the bicycle before it comes to rest

Let N = Required number of revolutions.

We know that distance travelled by the bicycle (x),

$197.2 = \pi DN = \pi \times 0.9N = 2.83N$

$N = 197.2 / 2.83 = 70$

Double Block or Shoe Brake

When a single block brake is applied to a rolling wheel, an additional load is thrown on the shaft bearings due to the normal force (RN). This produces bending of the shaft. In order to overcome this drawback, a double block or shoe brake, as shown in Fig. 19.9, is used. It consists of two brake blocks applied at the opposite ends of a diameter of the wheel which eliminate or reduces the unbalanced force on the shaft. The brake is set by a spring which pulls the upper ends of the brake arms together. When a force P is applied to the bell crank lever, the spring is compressed and the brake is released. This type of brake is often used on electric cranes and the force P is produced by an electromagnet or solenoid. When the current is switched off, there is no force on the bell crank lever and the brake is engaged automatically due to the spring force and thus there will be no downward movement to the load.

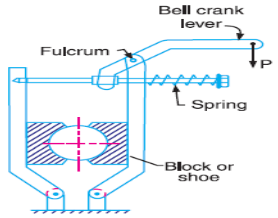


Fig. 19.9. Double block or shoe brake.

In a double block brake, the braking action is doubled by the use of two blocks and these blocks may be operated practically by the same force which will operate one. In case of double block or shoe brake, the braking torque is given by

$$T_B = (F_{t1} + F_{t2})\, r$$

Where F_{t1} and F_{t2} are the braking forces on the two blocks.

Internal Expanding Brake

An internal expanding brake consists of two shoes *S*1 and *S*2 as shown in Fig. 19.24. The outer surface of the shoes are lined with some friction material (usually with Ferodo) to increase the coefficient of friction and to prevent wearing away of the metal. Each shoe is pivoted at one end about a fixed fulcrum *O*1 and *O*2 and made to contact a cam at the other end. When the cam rotates, the shoes are pushed outwards against the rim of the drum. The friction between the shoes and the drum produces the braking torque and hence reduces the speed of the drum. The shoes are normally held in off position by a spring as shown in Fig. 19.24. The drum encloses the entire mechanism to keep out dust and moisture. This type of brake is commonly used in motor cars and light trucks.

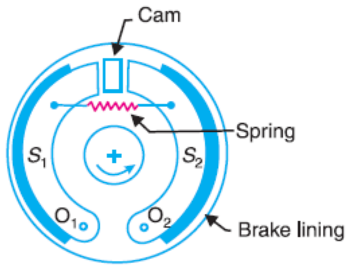


Fig. 19.24. Internal expanding brake.

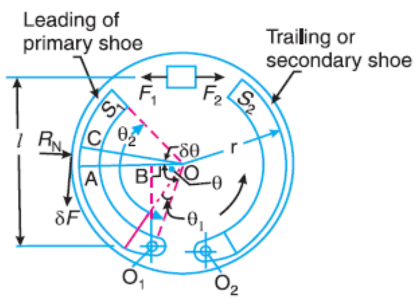


Fig. 19.25. Forces on an internal expanding brake.

We shall now consider the forces acting on such a brake, when the drum rotates in the anticlockwise direction as shown in Fig. 19.25. It may be noted that for the anticlockwise direction, the left hand shoe is known as ***leading*** or ***primary shoe*** while the right hand shoe is known as ***trailing*** or ***secondary shoe.***

Let
r = Internal radius of the wheel rim,
b = Width of the brake lining,
p_1 = Maximum intensity of normal pressure,
p_N = Normal pressure,
F_1 = Force exerted by the cam on the leading shoe, and
F_2 = Force exerted by the cam on the trailing shoe.

Consider a small element of the brake lining

AC subtending an angle δθ at the centre. Let *OA* makes an angle θ with OO_1 as shown in Fig. 19.25. It is assumed that the pressure distribution on the shoe is nearly uniform, however the friction lining wears out more at the free end. Since the shoe turns about*O*1, therefore the rate of wear of the shoe lining at *A* will be proportional to the radial displacement of that point. The rate of wear of the shoe lining varies directly as the perpendicular distance from $O_1$to *OA*, *i.e.*

O_1B. From the geometry of the figure,

$$O_1B = OO_1 \sin\theta$$

normal pressure at *A*,

$$p_N \propto \sin\theta \quad \text{or} \quad p_N = p_1 \sin\theta$$

∴ Normal force acting on the element,

$$\delta R_N = \text{Normal pressure} \times \text{Area of the element}$$

$$= p_N(b.r.\delta\theta) = p_1 \sin\theta(b.r.\delta\theta)$$

and braking or friction force on the element,

$$\delta F = \mu \times \delta R_N = \mu.p_1 \sin\theta(b.r.\delta\theta)$$

∴ Braking torque due to the element about *O*,

$$\delta T_B = \delta F \times r = \mu.p_1 \sin\theta(b.r.\delta\theta)r = \mu.p_1 b\,r^2(\sin\theta.\delta\theta)$$

and total braking torque about *O* for whole of one shoe,

$$T_B = \mu\, p_1 b\, r^2 \int_{\theta_1}^{\theta_2} \sin\theta\, d\theta = \mu\, p_1 b\, r^2 [-\cos\theta]_{\theta_1}^{\theta_2}$$

$$= \mu\, p_1 b r^2 (\cos\theta_1 - \cos\theta_2)$$

Moment of normal force δR_N of the element about the fulcrum O_1,

$$\delta M_N = \delta R_N \times O_1B = \delta R_N (OO_1 \sin\theta)$$

$$= p_1 \sin\theta(b.r.\delta\theta)(OO_1 \sin\theta) = p_1 \sin^2\theta(b.r.\delta\theta)OO_1$$

∴ Total moment of normal forces about the fulcrum O_1,

$$M_N = \int_{\theta_1}^{\theta_2} p_1 \sin^2\theta(b.r.\delta\theta)OO_1 = p_1.b.r.OO_1 \int_{\theta_1}^{\theta_2} \sin^2\theta\, d\theta$$

$$= p_1.b.r.OO_1 \int_{\theta_1}^{\theta_2} \frac{1}{2}(1-\cos 2\theta)d\theta \qquad \ldots\left[\because \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)\right]$$

$$= \frac{1}{2} p_1.b.r.OO_1 \left[\theta - \frac{\sin 2\theta}{2} \right]_{\theta_1}^{\theta_2}$$

$$= \frac{1}{2} p_1.b.r.OO_1 \left[\theta_2 - \frac{\sin 2\theta_2}{2} - \theta_1 + \frac{\sin 2\theta_1}{2} \right]$$

$$= \frac{1}{2} p_1.b.r.OO_1 \left[(\theta_2 - \theta_1) + \frac{1}{2}(\sin 2\theta_1 - \sin 2\theta_2) \right]$$

Moment of frictional force δF about the fulcrum O_1,

$$\delta M_F = \delta F \times AB = \delta F (r - OO_1 \cos\theta) \qquad \ldots (\because AB = r - OO_1 \cos\theta)$$

$$= \mu\, p_1 \sin\theta (b.r.\delta\theta)(r - OO_1 \cos\theta)$$

$$= \mu.p_1.b.r(r\sin\theta - OO_1 \sin\theta\cos\theta)\delta\theta$$

$$= \mu.p_1.b.r\left(r\sin\theta - \frac{OO_1}{2}\sin 2\theta \right)\delta\theta \qquad \ldots (\because 2\sin\theta\cos\theta = \sin 2\theta$$

$\therefore$ Total moment of frictional force about the fulcrum O_1,

$$M_F = \mu\, p_1 b\, r \int_{\theta_1}^{\theta_2} \left(r\sin\theta - \frac{OO_1}{2}\sin 2\theta \right) d\theta$$

$$= \mu\, p_1 b\, r \left[-r\cos\theta + \frac{OO_1}{4}\cos 2\theta \right]_{\theta_1}^{\theta_2}$$

$$= \mu\, p_1 b\, r \left[-r\cos\theta_2 + \frac{OO_1}{4}\cos 2\theta_2 + r\cos\theta_1 - \frac{OO_1}{4}\cos 2\theta_1 \right]$$

$$= \mu\, p_1 b\, r \left[r(\cos\theta_1 - \cos\theta_2) + \frac{OO_1}{4}(\cos 2\theta_2 - \cos 2\theta_1) \right]$$

Now for leading shoe, taking moments about the fulcrum O_1,

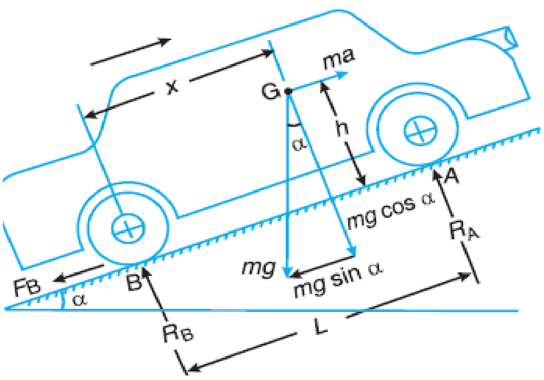
$$F_1 \times l = M_N - M_F$$

and for trailing shoe, taking moments about the fulcrum O_2,

$$F_2 \times l = M_N + M_F$$

Braking of a Vehicle

In a four wheeled moving vehicle, the brakes may be applied to
1. the rear wheels only,
2. the front wheels only, and
3. all the four wheels.
In all the above mentioned three types of braking, it is required to determine the retardation of the vehicle when brakes are applied. Since the vehicle retards, therefore it is a problem of dynamics. But it may be reduced to an equivalent problem of statics by including the inertia force in the system of forces actually applied to the vehicle. The inertia force is equal and opposite to the braking force causing retardation.
Now, consider a vehicle moving up an inclined plane, as shown in Fig.



Let α= Angle of inclination of the plane to the horizontal,
m = Mass of the vehicle in kg (such that its weight is $m.g$ newtons),
h = Height of the C.G. of the vehicle above the road surface in metres,
x = Perpendicular distance of C.G. from the rear axle in metres,
L = Distance between the centres of the rear and front wheels of the vehicle in metres,
R_A = Total normal reaction between the ground and the front wheels in newtons,
R_B = Total normal reaction between the ground and the rear wheels in newtons,
μ = Coefficient of friction between the tyres and road surface, and
a = Retardation of the vehicle in m/s2.

We shall now consider the above mentioned three cases of braking, one by one. In all these cases, the braking force acts in the opposite direction to the direction of motion of the vehicle.

1. When the brakes are applied to the rear wheels only

It is a common way of braking the vehicle in which the braking force acts at the rear wheels only.

Let F_B = Total braking force (in newtons) acting at the rear wheels due to the application of the brakes. Its maximum value is $\mu.R_B$.

The various forces acting on the vehicle are shown in Fig. For the equilibrium of the vehicle, the forces acting on the vehicle must be in equilibrium.
Resolving the forces parallel to the plane,

$$F_B + m.g.\sin\alpha = m.a \; . . . \; \textbf{(\textit{i})}$$

Resolving the forces perpendicular to the plane,

$$R_A + R_B = m\,g\cos\alpha \ldots\ldots\ldots \textbf{(\textit{ii})}$$

Taking moments about G, the centre of gravity of the vehicle

$$F_B \times h + R_B \times x = R_A\,(L - x) \ldots\ldots\; . . . \; \textbf{(\textit{iii})}$$

Substituting the value of $F_B = \mu.R_B$, and $R_A = m.g\cos\alpha - R_B$ [from equation **(*ii*)**] in the above expression, we have

$$\mu.R_B \times h + R_B \times x = (m.g\cos\alpha - R_B)\,(L - x)$$

$$R_B\,(L + \mu.h) = m.g\cos\alpha(L - x)$$

$$R_B = \frac{m.g\cos\alpha(L-x)}{L+\mu.h}$$

and $$R_A = m.g\cos\alpha - R_B = m.g\cos\alpha - \frac{m.g\cos\alpha(L-x)}{L+\mu.h}$$

$$= \frac{m.g\cos\alpha(x+\mu.h)}{L+\mu.h}$$

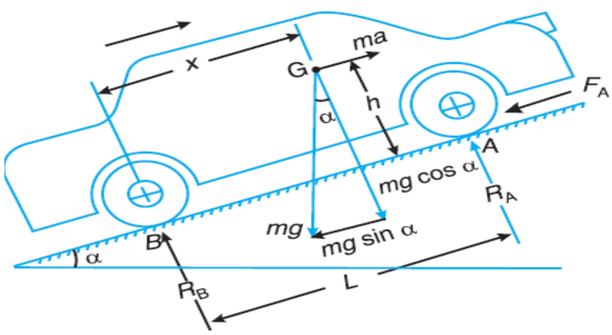
We know from equation (*i*),

$$a = \frac{F_B + m.g\sin\alpha}{m} = \frac{F_B}{m} + g\sin\alpha = \frac{\mu.R_B}{\text{m}} + g\sin\alpha$$

$$= \frac{\mu.g\cos\alpha(L-x)}{L+\mu.h} + g\sin\alpha \qquad \text{. . . (Substituting the value of } R_B\text{)}$$

2. When the brakes are applied to front wheels only

It is a very rare way of braking the vehicle, in which the braking force acts at the front wheels only.



Let F_A = Total braking force (in newtons) acting at the front wheels due to the application of brakes. Its maximum value is $\mu.R_A$.

The various forces acting on the vehicle are shown in Fig. Resolving the forces parallel to the plane,

$$F_A + m.g \sin\alpha = m.a \quad \ldots (\textit{i})$$

Resolving the forces perpendicular to the plane,

$$R_A + R_B = m.g \cos\alpha \quad \ldots (\textit{ii})$$

Taking moments about G, the centre of gravity of the vehicle,

$$F_A \times h + R_B \times x = R_A (L - x)$$

Substituting the value of $F_A = \mu.R_A$ and $R_B = m.g \cos\alpha - R_A$ [from equation **(*ii*)**] in the above expression, we have

$$\mu.R_A \times h + (m.g \cos\alpha - R_A)\, x = R_A (L - x)$$

$$\mu.R_A \times h + m.g \cos\alpha \times x = R_A \times L$$

$$\therefore \qquad R_A = \frac{m.g \cos\alpha \times x}{L - \mu.h}$$

and

$$R_B = m.g \cos\alpha - R_A = m.g \cos\alpha - \frac{m.g \cos\alpha \times x}{L - \mu.h}$$

$$= m.g \cos\alpha \left(1 - \frac{x}{L - \mu.h}\right) = m.g \cos\alpha \left(\frac{L - \mu.h - x}{L - \mu.h}\right)$$

We know from equation (*i*),

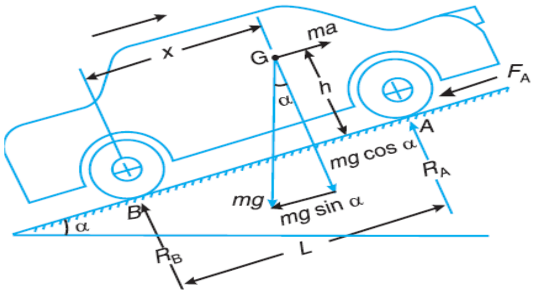
$$a = \frac{F_A + m.g \sin\alpha}{m} = \frac{\mu.R_A + m.g \sin\alpha}{m}$$

$$= \frac{\mu.m.g \cos\alpha \times x}{(L - \mu.h)m} + \frac{m.g \sin\alpha}{m} \qquad \ldots (\text{Substituting the value of } R_A)$$

$$= \frac{\mu.g \cos\alpha \times x}{L - \mu.h} + g \sin\alpha$$

3. When the brakes are applied to all the four wheels

This is the most common way of braking the vehicle, in which the braking force acts on both the rear and front wheels.



Let F_A = Braking force provided by the front wheels = $\mu.R_A$, and

F_B = Braking force provided by the rear wheels = $\mu.R_B$.

Little consideration will show that when the brakes are applied to all the four wheels, the braking distance (i.e. the distance in which the vehicle is brought to rest after applying the brakes) will be the least. It is due to this reason that the brakes are applied to all the four wheels. The various forces acting on the vehicle are shown in fig.

Resolving the forces parallel to the plane,

$F_A + F_B + m.g\sin\alpha = m.a$…….(i)

Resolving forces vertical to the plane

$$R_A + R_B = m.g \cos\alpha . . . \textbf{(\textit{ii})}$$

Taking moments about *G*, the centre of gravity of the vehicle,

$(F_A + F_B)\, h + R_B \times x = R_A(L - x)$…….(iii)

Substituting the value of $F_A = \mu.R_A$, $F_B = \mu.R_B$ and $R_B = m.g \cos\alpha - R_A$

[From equation **(*ii*)**] in the above expression,

$\mu\,(R_A + R_B)\, h + (m\, g\cos\alpha - R_A)\, x = R_A(L - x)$

$\mu\,(R_A + mg \cos\alpha - R_A)h + (mg \cos \alpha - R_A)x = R_A(L - x)$

$\mu.m.g \cos\alpha \times h + m.g \cos\alpha \times x = R_A \times L$

$$R_A = \frac{m.g\cos\alpha(\mu.h+x)}{L}$$

$$R_B = m.g\cos\alpha - R_A = m.g\cos\alpha - \frac{mg\cos\alpha(\mu.h+x)}{L}$$

$$= m.g\cos\alpha\left[1-\frac{\mu.h+x}{L}\right] = m.g\cos\alpha\left(\frac{L-\mu.h-x}{L}\right)$$

Now from equation (i),

$\mu.R_A + \mu R_B + m.g \sin\alpha = m.a$

$\mu(R_A + R_B) + m.g \sin\alpha = m.a$

$\mu.m.g.\cos\alpha + m.g \sin\alpha = m.a$ . . . [From equation (ii)]

$a = g(\mu.\cos\alpha + \sin\alpha)$

PROBLEMS

Example 1. A car moving on a level road at a speed 50 km/h has a wheel base 2.8metres, distance of C.G. from ground level 600 mm, and the distance of C.G. from rear wheels 1.2metres. Find the distance travelled by the car before coming to rest when brakes are applied,

1. To the rear wheels,
2. To the front wheels, and
3. To all the four wheels. The coefficient of friction between the tyres and the road may be taken as 0.6.

Solution.

Given : u = 50 km/h = 13.89 m/s ; L = 2.8 m ; h = 600 mm = 0.6 m ; x = 1.2 m ; μ = 0.6

Let s = Distance travelled by the car before coming to rest.

1. **When brakes are applied to the rear wheels**

Since the vehicle moves on a level road, therefore retardation of the car,

$$a = \frac{\mu.g(L-x)}{L+\mu.h} = \frac{0.6\times 9.81(2.8-1.2)}{2.8+0.6\times 0.6} = 2.98 \text{ m/s}^2$$

We know that for uniform retardation,

$$s = \frac{u^2}{2a} = \frac{(13.89)^2}{2\times 2.98} = 32.4 \text{ m}$$

2. **When brakes are applied to the front wheels**

Since the vehicle moves on a level road, therefore retardation of the car,

$$a = \frac{\mu.g.x}{L-\mu.h} = \frac{0.6\times 9.18\times 1.2}{2.8-0.6\times 0.6} = 2.9 \text{ m/s}^2$$

We know that for uniform retardation,

$$s = \frac{u^2}{2a} = \frac{(13.89)^2}{2\times 2.9} = 33.26 \text{ m}$$

3. **When the brakes are applied to all the four wheels**

Since the vehicle moves on a level road, therefore retardation of the car,
$a = g.\mu = 9.81\times0.6 = 5.886$ m/s2

We know that for uniform retardation,

$$s = \frac{u^2}{2a} = \frac{(13.89)^2}{2\times 5.886} = 16.4 \text{ m}$$

Example2. A vehicle moving on a rough plane inclined at 10° with the horizontal at a speed of 36 km/h has a wheel base 1.8 metres. The centre of gravity of the vehicle is 0.8 metre from the rear wheels and 0.9 metre above the inclined plane. Find the distance travelled by the vehicle before coming to rest and the time taken to do so when
1. The vehicle moves up the plane, and
2. The vehicle moves down the plane.
The brakes are applied to all the four wheels and the coefficient of friction is 0.5.

Solution.
Given : $\alpha = 10°$; u = 36 km / h = 10 m / s ; L = 1.8 m ; x = 0.8 m ; h = 0.9 m ; $\mu = 0.5$
Let s = Distance travelled by the vehicle before coming to rest, and

t = Time taken by the vehicle in coming to rest.

1. **When the vehicle moves up the plane and brakes are applied to all the four wheel**

Since the vehicle moves up the inclined plane, therefore retardation of the vehicle,

$a = g\,(\mu \cos\alpha + \sin\alpha)$

$= 9.81\,(0.5\cos10° + \sin10°) = 9.81(0.5\times0.9848 + 0.1736) = 6.53$ m/s^2

We know that for uniform retardation,

$$s = \frac{u^2}{2a} = \frac{(10)^2}{2\times 6.53} = 7.657 \text{ m}$$

and final velocity of the vehicle (v),

$0 = u + a.t = 10 - 6.53\,t$ ……….(Minus sign due to retardation)

$t = 10 / 6.53 = 1.53$

2. **When the vehicle moves down the plane and brakes are applied to all the four wheels**

   Since the vehicle moves down the inclined plane, therefore retardation of the vehicle,

   $a = g\ (\mu \cos\alpha - \sin \alpha)$

   $= 9.81(0.5\cos 10° - \sin 10°) = 9.81(0.5 \times 0.9848 - 0.1736) = 3.13\ m/s^2$

   We know that for uniform retardation,

$$s = \frac{u^2}{2a} = \frac{(10)^2}{2 \times 3.13} = 16\ \text{m}$$

and final velocity of the vehicle (v),

$0 = u + a.t = 10 - 3.13\ t$ . . . (Minus sign due to retardation)

$t = 10/3.13 = 3.2$ s

DYNAMOMETER

A dynamometer is a brake but in addition it has a device to measure the frictional resistance. Knowing the frictional resistance, we may obtain the torque transmitted and hence the power of the engine.

Types of Dynamometers

1. Absorption dynamometers,
2. Transmission dynamometers.

In the *absorption* ***dynamometers,*** the entire energy or power produced by the engine is absorbed by the friction resistances of the brake and is transformed into heat, during the process of measurement. But in the ***transmission dynamometers,*** the energy is not wasted in friction but is used for doing work. The energy or power produced by the engine is transmitted through the dynamometer to some other machines where the power developed is suitably measured.

Classification of Absorption Dynamometers

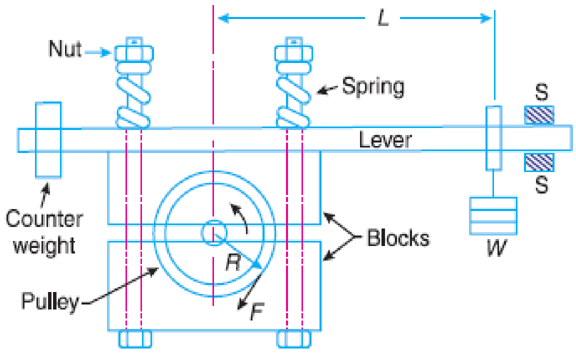
1. Prony brake dynamometer,

2.Rope brake dynamometer.

1. **Prony brake dynamometer**

A simplest form of an absorption type dynamometer is a prony brake dynamometer, as shown in Fig. It consists of two wooden blocks placed around a pulley fixed to the shaft of an engine whose power is required to be measured. The blocks are clamped by means of two bolts and nuts, as shown in Fig.

A helical spring is provided between the nut and the upper block to adjust the pressure on the pulley to control its speed. The upper block has a long lever attached to it and carries a weight *W* at its outer end. A counter weight is placed at the other end of the lever which balances the brake when unloaded. Two stops S, S are provided to limit the motion of the lever.



When the brake is to be put in operation, the long end of the lever is loaded with suitable weights *W* and the nuts are tightened until the engine shaft runs at a constant speed and the lever is in horizontal position. Under these conditions, the moment due to the weight *W* must balance the moment of the frictional resistance between the blocks and the pulley.

Let W = Weight at the outer end of the lever in newtons,
L = Horizontal distance of the weight W from the centre of the pulley in metres,
F = Frictional resistance between the blocks and the pulley in newtons,
R = Radius of the pulley in metres, and
N = Speed of the shaft in r.p.m.

We know that the moment of the frictional resistance or torque on the shaft,

$$T = W.L = F.R \text{ N-m}$$

Work done in one revolution

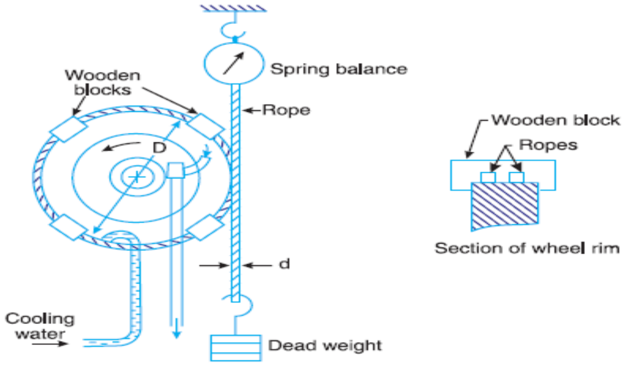
Work done in one revolution
= Torque × Angle turned in radians
= $T \times 2\pi$ N-m
_ Work done per minute

= $T \times 2\pi N$ N-m
We know that brake power of the engine

$$B.P. = \frac{\text{Work done per min.}}{60} = \frac{T \times 2\pi N}{60} = \frac{W.L \times 2\pi\ N}{60} \text{ watts}$$

Rope Brake Dynamometer

It is another form of absorption type dynamometer which is most commonly used for measuring the brake power of the engine. It consists of one, two or more ropes wound around the flywheel or rim of a pulley fixed rigidly to the shaft of an engine. The upper end of the ropes is attached to a spring balance while the lower end of the ropes is kept in position by applying a dead weight as shown in Fig.19.32. In order to prevent the slipping of the rope over the flywheel, wooden blocks are placed at intervals around the circumference of the flywheel.



In the operation of the brake, the engine is made to run at a constant speed. The frictional torque, due to the rope, must be equal to the torque being transmitted by the engine.

Let W = Dead load in newtons,

S = Spring balance reading in newtons,

D = Diameter of the wheel in metres,

d = diameter of rope in metres, and

N = Speed of the engine shaft in r.p.m.

Net load on the brake= $(W - S)$ N

We know that distance moved in one revolution= $\pi (D + d)$ m

Work done per revolution= $(W - S)$ _ $(D + d)$ N-m

Work done per minute= $(W - S)\ \pi\ (D + d)\ N$ N-m

Brake power of the engine,

$$\text{B.P} = \frac{\text{Work done per min}}{60} = \frac{(W - S)\ \pi\ (D + d) N}{60} \text{ watts}$$

,

If the diameter of the rope (d) is neglected, then brake power of the engine

$$\text{B.P.} = \frac{(W - S)\ \pi D\ N}{60} \text{ watts}$$

Example 1. In a laboratory experiment, the following data were recorded with rope brake: Diameter of the flywheel 1.2 m; diameter of the rope12.5 mm; speed of the engine 200 r.p.m.; dead load on the brake 600 N; spring balance reading 150 N. Calculate the brake power of the engine.

Solution. Given : D = 1.2 m ; d = 12.5 mm
= 0.0125 m ; N = 200 r.p.m ; W = 600 N ; S = 150 N
We know that brake power of the engine,

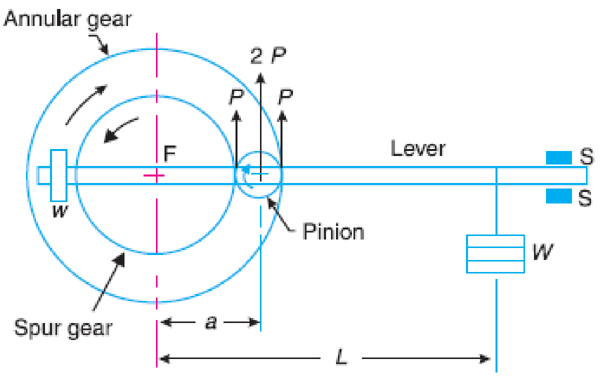
$$\text{B.P.} = \frac{(W - S)\,\pi\,(D + d)N}{60} = \frac{(600 - 150)\,\pi\,(1.2 + 0.0125)200}{60} = 5715 \text{ W}$$

Classification of Transmission Dynamometers

1. Epicyclic-train dynamometer,
2. Belt transmission dynamometer, and
3. Torsion dynamometer

Epicyclic-train Dynamometer

An epicyclic-train dynamometer, as shown in Fig. 19.33, consists of a simple epicyclic train of gears, *i.e.* a spur gear, an annular gear (a gear having internal teeth) and a pinion. The spur gear is keyed to the engine shaft (*i.e.* driving shaft) and rotates in anticlockwise direction. The annular gear is also keyed to the driving shaft and rotates in clockwise direction. The pinion or the intermediate gear meshes with both the spur and annular gears. The pinion revolves freely on a lever which is pivoted to the common axis of the driving and driven shafts. A weight *w* is placed at the smaller end of the lever in order to keep it in position. A little consideration will show that if the friction of the pinon which the pinion rotates is neglected, then the tangential effort *P* exerted by the spur gear on the pinion and the tangential reaction of the annular gear on the pinion are equal.



Since these efforts act in the upward direction as shown, therefore total upward force on the lever acting through the axis of the pinion is 2*P*. This force tends to rotate the lever about its fulcrum and it is balanced by a dead weight *W* at the end of the lever. The stops *S*, *S* are provided to control the movement of the lever.

For equilibrium of the lever, taking moments about the fulcrum *F*,

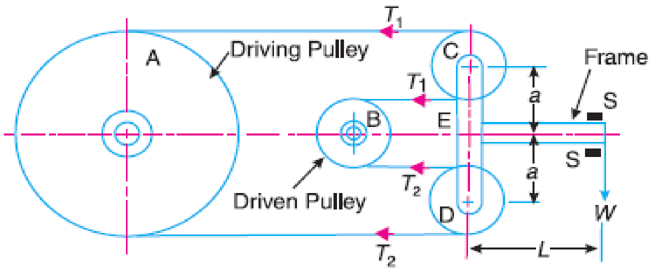
$2P \times a = W.L$ or $P = W.L/2a$
R = Pitch circle radius of the spur gear in metres, and
N = Speed of the engine shaft in r.p.m.

Torque transmitted, $T = P.R$

power transmitted $= \frac{T \times 2\pi N}{60} = \frac{P.R \times 2\pi N}{60}$ watts

Belt Transmission Dynamometer-Froude or Throneycroft Transmission Dynamometer

When the belt is transmitting power from one pulley to another, the tangential effort on the driven pulley is equal to the difference between the tensions in the tight and slack sides of the belt. A belt dynamometer is introduced to measure directly the difference between the tensions of the belt, while it is running.



A belt transmission dynamometer, as shown in Fig, is called a Froude or Throneycroft transmission dynamometer. It consists of a pulley *A* (called driving pulley) which is rigidly fixed to the shaft of an engine whose power is required to be measured. There is another pulley *B* (called driven pulley) mounted on another shaft to which the power from pulley *A* is transmitted. The pulleys *A* and *B* are connected by means of a continuous belt passing round the two loose pulleys *C* and *D* which are mounted on a *T*-shaped frame. The frame is pivoted at *E* and its movement is controlled by two stops *S*,*S*. Since the tension in the tight side of the belt ($T1$) is greater than the tension in the slack side of the belt ($T2$), therefore the total force acting on the pulley *C* (*i.e.* $2T1$) is greater than the total force acting on the pulley *D* (*i.e.* $2T2$). It is thus obvious that the frame causes movement about *E* in the anticlockwise direction. In order to balance it, a weight *W* is applied at a distance *L* from *E* on the frame as shown in Fig.

Now taking moments about the pivot *E*, neglecting friction,

$2T_1 \times a = 2T_2 \times a + W L$ $\qquad T_1 - T_2 = \frac{W.L}{2a}$

Let D = diameter of the pulley A in metres,
N = Speed of the engine shaft in r.p.m.
Work done in one revolution $= (T_1 - T_2)\pi D$ N-m

work done per minute $= (T_1 - T_2)\pi DN$ N-m

$\therefore$ Brake power of the engine, $B.P. = \frac{(T_1 - T_2)\pi DN}{60}$ watts

Torsion Dynamometer

A torsion dynamometer is used for measuring large powers particularly the power transmitted along the propeller shaft of a turbine or motor vessel. A little consideration will show that when the power is being transmitted, then the driving end of the shaft twists through a small angle relative to the driven end of the shaft. The amount of twist depends upon many factors such as torque acting on the shaft (*T*), length of the shaft (*l*), diameter of the shaft (*D*) and modulus of rigidity (*C*) of the material of the shaft. We know that the torsion equation is

$$\frac{T}{J} = \frac{C.\theta}{l}$$

where θ = Angle of twist in radians, and

J = Polar moment of inertia of the shaft.

For a solid shaft of diameter *D*, the polar moment of inertia

$$J = \frac{\pi}{32} \times D^4$$

and for a hollow shaft of external diameter *D* and internal diameter *d*, the polar moment of inertia,

$$J = \frac{\pi}{32}(D^4 - d^4)$$

From the above torsion equation,

$$T = \frac{C.J}{l} \times \theta = k.\theta$$

where $k = C.J/l$ is a constant for a particular shaft. Thus, the torque acting on the shaft is proportional to the angle of twist. This means that if the angle of twist is measured by some means, then the torque and hence the power transmitted may be determined.

We know that the power transmitted

P= 2πNT/60watts,

where *N* is the speed in r.p.m.

PROBLEMS

Example 1. A torsion dynamometer is fitted to a propeller shaft of a marine engine. It is found that the shaft twists 2° in a length of 20 metres at 120 r.p.m. If the shaft is hollow with 400 mm external diameter and 300 mm internal diameter, find the power of the engine. Take modulus of rigidity for the shaft material as 80 GPa.

Solution.

Given : _ = 2° = 2 × _ /180 = 0.035 rad ; *l* = 20 m ; *N* = 120 r.p.m. ; *D* = 400 mm= 0.4 m ; *d* = 300 mm = 0.3 m ; *C* = 80 GPa = 80 × 109 N/m^2

We know that polar moment of inertia of the shaft

$$J = \frac{\pi}{32}(D^4 - d^4) = \frac{\pi}{32}\left[(0.4)^4 - (0.3)^4\right] = 0.0017\,\text{m}^4$$

and torque applied to the shaft,

$$T = \frac{C.J}{l} \times \theta = \frac{80 \times 10^9 \times 0.0017}{20} \times 0.035 = 238 \times 10^3 \text{ N-m}$$

We know that power of the engine,

$$P = \frac{T \times 2\pi N}{60} = \frac{238 \times 10^3 \times 2\pi \times 120}{60} = 2990 \times 10^3 \text{ W} = 2990 \text{ kW}$$

UNIT-IV
BALANCING

Balancing of Rotating Masses

The high speed of engines and other machines is a common phenomenon now-a-days. It is, therefore, very essential that all the rotating and reciprocating parts should be completely balanced as far as possible. If these parts are not properly balanced, the dynamic forces are set up. These forces not only increase the loads on bearings and stresses in the various members, but also produce unpleasant and even dangerous vibrations. In this chapter we shall discuss the balancing of unbalanced forces caused by rotating masses, in order to minimize pressure on the main bearings when an engine is running.

We have already discussed, that whenever a certain mass is attached to a rotating shaft, it exerts some centrifugal force, whose effect is to bend the shaft and to produce vibrations in it. In order to prevent the effect of centrifugal force, another mass is attached to the opposite side of the shaft, at such a position so as to balance the effect of the centrifugal force of the first mass. This is done in such a way that the centrifugal force of both the masses are made to be equal and opposite. The process of providing the second mass in order to counteract the effect of the centrifugal force of the first mass, is called ***balancing of rotating masses*.**

The following cases are important from the subject point of view:

1. Balancing of a single rotating mass by a single mass rotating in the same plane.
2. Balancing of a single rotating mass by two masses rotating in different planes.
3. Balancing of different masses rotating in the same plane.
4. Balancing of different masses rotating in different planes.

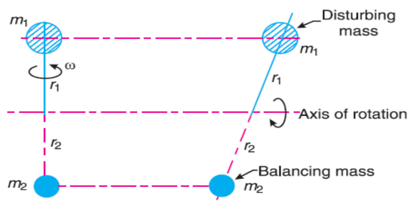
We shall now discuss these cases, in detail, in the following pages.

Balancing of a Single Rotating Mass By a Single Mass Rotating in the Same Plane

Consider a disturbing mass m_1 attached to a shaft rotating at ω rad/s as shown in Fig. Let r_1 be the radius of rotation of the mass m_1 (*i.e.* distance between the axis of rotation of the shaft and the centre of gravity of the mass m_1).We know that the centrifugal force exerted by the mass m_1 on the shaft,

$$F_{C1} = m_1 \cdot \omega^2 \cdot r_1$$

This centrifugal force acts radially outwards and thus produces bending moment on the shaft. In order to counteract the effect of this force, a balancing mass (m_2) may be attached in the same plane of rotation as that of disturbing mass (m_1) such that the centrifugal forces due to the two masses are equal and opposite



Let $r2$ = Radius of rotation of the balancing mass $m2$ (*i.e.* distance between the axis of rotation of the shaft and the centre of gravity of mass m_2).

∴ Centrifugal force due to mass m_2

$$F_{C2} = m_2 \cdot \omega^2 \cdot r_2$$

Equating equations ***(i)*** and ***(ii)*,**

$$m_1.\omega^2 \cdot r_1 = m_2 \cdot \omega^2 \cdot r_2 \quad \text{or} \quad m_1 \cdot r_1 = m_2 \cdot r_2$$

Balancing of a Single Rotating Mass By Two Masses Rotating in Different Planes

We have discussed in the previous article that by introducing a single balancing mass in the same plane of rotation as that of disturbing mass, the centrifugal forces are balanced. In other words, the two forces are equal in magnitude and opposite in direction. But this type of arrangement for balancing gives rise to a couple which tends to rock the shaft in its bearings. Therefore in order to put the system in complete balance, two balancing masses are placed in two different planes, parallel to the plane of rotation of the disturbing mass, in such a way that they satisfy the following two conditions of equilibrium.

1. The net dynamic force acting on the shaft is equal to zero. This requires that the line of action of three centrifugal forces must be the same. In other words, the centre of the masses of the system must lie on the axis of rotation. This is the condition for ***static balancing*.**

2. The net couple due to the dynamic forces acting on the shaft is equal to zero. In other words, the algebraic sum of the moments about any point in the plane must be zero.

The conditions **(1)** and **(2)**together give ***dynamic balancing*.** The following two possibilities may arise while attaching the two balancing masses:

1. The plane of the disturbing mass may be in between the planes of the two balancing masses, and

2. The plane of the disturbing mass may lie on the left or right of the two planes containing the balancing masses.

We shall now discuss both the above cases one by one

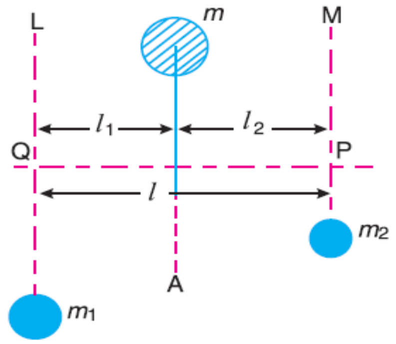
1. When the plane of the disturbing mass lies in between the planes of the two balancing masses

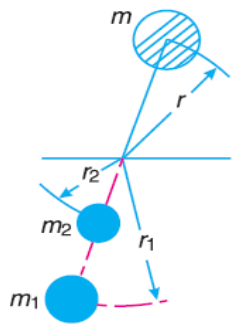
Consider a disturbing mass m lying in a plane A to be balanced by two rotating masses m_1 and m_2 lying in two different planes L and M as shown in Fig. Let r, r_1 and r_2 be the radii of rotation of the masses in planes A, L and M respectively.

Let l_1 = Distance between the planes A and L,

l_2 = Distance between the planes A and M, and

l = Distance between the planes L and M





We know that the centrifugal force exerted by the mass m in the plane A,

$$F_C = m \cdot \omega^2 \cdot r$$

Similarly, the centrifugal force exerted by the mass m_1 in the plane L,

$$F_{C1} = m_1 \cdot \omega^2 \cdot r_1$$

and, the centrifugal force exerted by the mass m_2 in the plane M,

$$F_{C2} = m_2 \cdot \omega^2 \cdot r_2$$

Since the net force acting on the shaft must be equal to zero, therefore the centrifugal force on the disturbing mass must be equal to the sum of the centrifugal forces on the balancing masses, therefore

$$F_C = F_{C1} + F_{C2} \quad \text{or} \quad m \cdot \omega^2 \cdot r = m_1 . \omega^2 \cdot r_1 + m_2 \cdot \omega^2 \cdot r_2$$

$$m \cdot r = m_1 \cdot r_1 + m_2 \cdot r_2 \qquad \ldots (i)$$

Now in order to find the magnitude of balancing force in the plane L (or the dynamic force at the bearing Q of a shaft), take moments about P which is the point of intersection of the plane M and the axis of rotation. Therefore

$$F_{C1} \times l = F_C \times l_2 \quad \text{or} \quad m_1 \cdot \omega^2 \cdot r_1 \times l = m \cdot \omega^2 \cdot r \times l_2$$

$$m_1 \cdot r_1 \cdot l = m \cdot r \cdot l_2 \quad \text{or} \quad m_1 \cdot r_1 = m \cdot r \times \frac{l_2}{l}$$

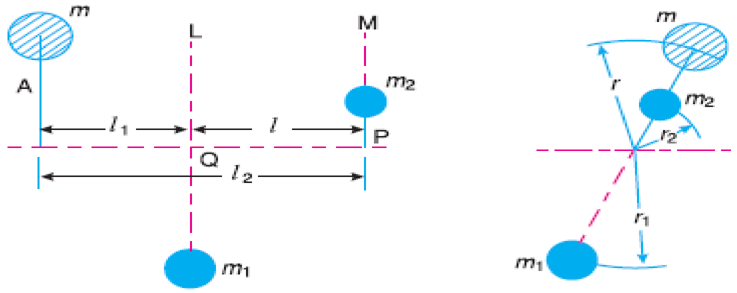
Similarly, in order to find the balancing force in plane M (or the dynamic force at the bearing P of a shaft), take moments about Q which is the point of intersection of the plane L and the axis of rotation. Therefore

$$F_{C2} \times l = F_C \times l_1 \quad \text{or} \quad m_2 \cdot \omega^2 \cdot r_2 \times l = m \cdot \omega^2 \cdot r \times l_1$$

$$m_2 \cdot r_2 \cdot l = m \cdot r \cdot l_1 \quad \text{or} \quad m_2 \cdot r_2 = m \cdot r \times \frac{l_1}{l}$$

It may be noted that equation **(*i*)** represents the condition for static balance, but in order to achieve dynamic balance, equations **(*ii*)** or **(*iii*)** must also be satisfied.

2. When the plane of the disturbing mass lies on one end of the planes of the balancing masses



In this case, the mass m lies in the plane A and the balancing masses lie in the planes L and M, as shown in Fig. As discussed above, the following conditions must be satisfied in order to balance the system, i.e.

$$F_C + F_{C2} = F_{C1} \quad \text{or} \quad m \cdot \omega^2 \cdot r + m_2 \cdot \omega^2 \cdot r_2 = m_1 \cdot \omega^2 \cdot r_1$$

$$m \cdot r + m_2 \cdot r_2 = m_1 . r_1 \qquad \ldots (iv)$$

Now, to find the balancing force in the plane *L* (or the dynamic force at the bearing *Q* of a shaft), take moments about *P* which is the point of intersection of the plane *M* and the axis of rotation. Therefore

$$F_{C1} \times l = F_C \times l_2 \quad \text{or} \quad m_1 \cdot \omega^2 \cdot r_1 \times l = m \cdot \omega^2 \cdot r \times l_2$$

$$m_1 \cdot r_1 \cdot l = m \cdot r \cdot l_2 \quad \text{or} \quad m_1 \cdot r_1 = m \cdot r \times \frac{l_2}{l} \qquad \ldots (v)$$

Similarly, to find the balancing force in the plane *M* (or the dynamic force at the bearing *P* of a shaft), take moments about *Q* which is the point of intersection of the plane *L* and the axis of rotation. Therefore

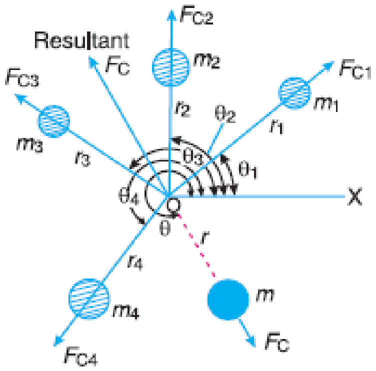
$$F_{C2} \times l = F_C \times l_1 \quad \text{or} \quad m_2 \cdot \omega^2 \cdot r_2 \times l = m \cdot \omega^2 \cdot r \times l_1$$

$$m_2 \cdot r_2 \cdot l = m \cdot r \cdot l_1 \quad \text{or} \quad m_2 \cdot r_2 = m \cdot r \times \frac{l_1}{l} \qquad \ldots (vi)$$

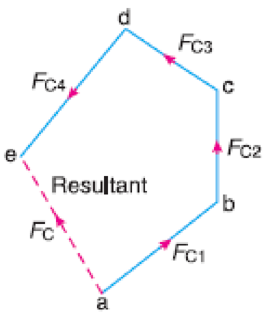
Balancing of Several Masses Rotating in the Same Plane

Consider any number of masses (say four) of magnitude m_1, m_2, m_3 and m_4 at distances of r_1, r_2, r_3 and r_4 from the axis of the rotating shaft. Let θ_1, θ_2, θ_3 and θ_4 be the angles of these masses with the horizontal line *OX*, as shown in Fig. Let these masses rotate about an axis through *O* and perpendicular to the plane of paper, with a constant angular velocity of ω rad/s.

The magnitude and position of the balancing mass may be found out analytically or graphically as discussed below :



(a) Space diagram.



(b) Vector diagram.

Analytical method

The magnitude and direction of the balancing mass may be obtained, analytically, as discussed below :

1. First of all, find out the centrifugal force*(or the product of the mass and its radius of rotation) exerted by each mass on the rotating shaft.
2. Resolve the centrifugal forces horizontally and vertically and find their sums, i.e. ΣH and ΣV .We know that Sum of horizontal components of the centrifugal forces,

$$\Sigma H = m_1 \cdot r_1 \cos \theta_1 + m_2 \cdot r_2 \cos \theta_2 + \ldots\ldots$$

sum of vertical components of the centrifugal forces,

$$\Sigma V = m_1 \cdot r_1 \sin \theta_1 + m_2 \cdot r_2 \sin \theta_2 + \ldots\ldots$$

3. Magnitude of the resultant centrifugal force,

$$F_C = \sqrt{(\Sigma H)^2 + (\Sigma V)^2}$$

4.If θ is the angle, which the resultant force makes with the horizontal, then

$$\tan \theta = \Sigma V / \Sigma H$$

5.The balancing force is then equal to the resultant force, but in **opposite direction.**

6.Now find out the magnitude of the balancing mass, such that

FC = m·r where m = Balancing mass, and

r = Its radius of rotation.

Graphical method

The magnitude and position of the balancing mass may also be obtained graphically as discussed below :

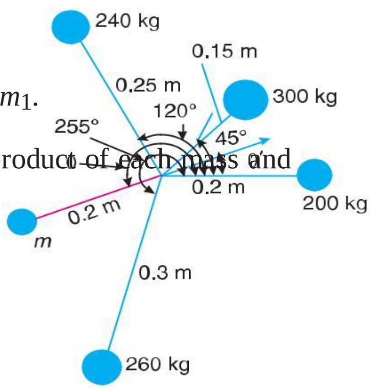
1. First of all, draw the space diagram with the positions of the several masses,
2. Find out the centrifugal force (or product of the mass and radius of rotation) exerted by each mass on the rotating shaft.
3. Now draw the vector diagram with the obtained centrifugal forces (or the product of the masses and their radii of rotation), such that ab represents the centrifugal force exerted by the mass m_1 (or $m_1.r_1$) in magnitude and direction to some suitable scale. Similarly, draw bc, cd and de to represent centrifugal forces of other masses m_2, m_3 and m_4 (or $m_2.r_2$, $m_3.r_3$ and $m_4.r_4$).
4. Now, as per polygon law of forces, the closing side ae represents the resultant force in magnitude and direction.
5. The balancing force is, then, equal to the resultant force, but in **opposite direction.**
6. Now find out the magnitude of the balancing mass (m) at a given radius of rotation (r), such that

$m \cdot \omega^2 \cdot r$ = Resultant centrifugal force

$m.r$ = Resultant of $m_1.r_1$, $m_2.r_2$, $m_3.r_3$ and $m_4.r_4$

Example 1. Four masses m_1, m_2, m_3 and m_4 are 200 kg, 300 kg, 240 kg and 260 kg respectively. The corresponding radii of rotation are 0.2 m, 0.15 m, 0.25 m and 0.3 m respectively and the angles between successive masses are 45°, 75° and 135°. Find the position and magnitude of the balance mass required, if its radius of rotation is 0.2 m.

Solution. Given : m_1 = 200 kg ; m_2 = 300 kg ; m_3 = 240 kg ; m_4 = 260 kg ; r_1 = 0.2 m ; r_2 = 0.15 m ; r_3 = 0.25 m ; r_4 = 0.3 m ; θ_1 = 0° ; θ_2 = 45° ; θ_3 = 45° + 75° = 120° ; θ_4 = 45° + 75° + 135° = 255° ; r = 0.2 m



Let m = Balancing mass, and

F = The angle which the balancing mass makes with m_1.

Since the magnitude of centrifugal forces are proportional to the product of each mass and its radius, therefore

$m_1 \cdot r_1$ = 200 · 0.2 = 40 kg-m

$m_2 \cdot r_2$ = 300 · 0.15 = 45 kg-m

$m_3 \cdot r_3$ = 240 · 0.25 = 60 kg-m

$m_4 \cdot r_4$ = 260 · 0.3 = 78 kg-m

The problem may, now, be solved either analytically or graphically. But we shall solve the problem by both the methods one by one.

Analytical method

Resolving $m_1.r_1$, $m_2.r_2$, $m_3.r_3$ and $m_4.r_4$ horizontally,

$$\Sigma H = m_1 \cdot r_1 \cos\theta_1 + m_2 \cdot r_2 \cos\theta_2 + m_3 \cdot r_3 \cos\theta_3 + m_4 \cdot r_4 \cos\theta_4$$

$$= 40 \cos 0° + 45 \cos 45° + 60 \cos 120° + 78 \cos 255°$$

$$= 40 + 31.8 - 30 - 20.2 = 21.6 \text{ kg-m}$$

Now resolving vertically,

$$\Sigma V = m_1 \cdot r_1 \sin\theta_1 + m_2 \cdot r_2$$

$\sin\theta 2 + m3 \cdot r3 \sin\theta 3 + m4 \cdot r4 \sin\theta 4$

$$= 40 \sin 0° + 45 \sin 45° + 60 \sin 120° + 78 \sin 255°$$

$$= 0 + 31.8 + 52 - 75.3 = 8.5 \text{ kg-m}$$

$$\therefore \text{Resultant}, \quad R = \sqrt{(\Sigma H)^2 + (\Sigma V)^2} = \sqrt{(21.6)^2 + (8.5)^2} = 23.2 \text{ kg-m}$$

$$m \cdot r = R = 23.2 \quad \text{or} \quad m = 23.2 / r = 23.2 / 0.2 = 116 \text{ kg}$$

Since θ' is the angle of the resultant R from the horizontal mass of 200 kg, therefore the angle of the balancing mass from the horizontal mass of 200 kg,

$$= 180° + 21.48° = 201.48° \textbf{ Ans.}$$

2. Graphical method

The magnitude and the position of the balancing mass may also be found graphically as discussed below :

1.First of all, draw the space diagram showing the positions of all the given masses as shown

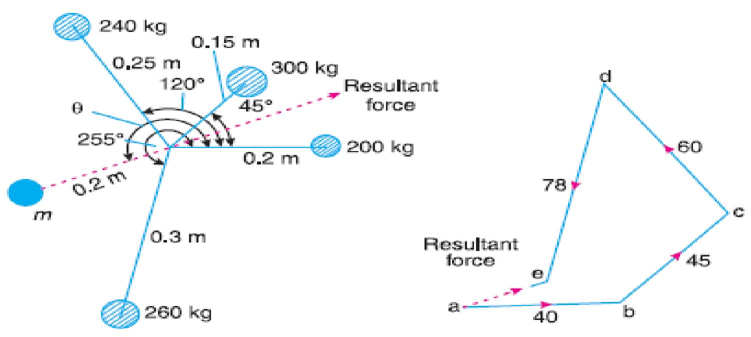
2.Since the centrifugal force of each mass is proportional to the product of the mass and radius, therefore

$$m_1.r_1 = 200 \times 0.2 = 40 \text{ kg-m}$$

$$m_2.r_2 = 300 \times 0.15 = 45 \text{ kg-m}$$

$$m_3.r_3 = 240 \times 0.25 = 60 \text{ kg-m}$$

$$m_4.r_4 = 260 \times 0.3 = 78 \text{ kg-m}$$



(a) Space diagram. (b) Vector diagram

4. The balancing force is equal to the resultant force, but **opposite** in direction. Since the balancing force is proportional to m.r, therefore

$$m \times 0.2 = \text{vector } ea = 23 \text{ kg-m} \quad \text{or} \quad m = 23/0.2 = \mathbf{115\ kg}$$

By measurement we also find that the angle of inclination of the balancing mass (m) from the horizontal mass of 200 kg,

$$\theta = 201°$$

θ

Balancing of Several Masses Rotating in Different Planes

When several masses revolve in different planes, they may be transferred to a **reference plane** (briefly written as **R.P.**), which may be defined as the plane passing through a point on the axis of rotation and perpendicular to it. The effect of transferring a revolving mass (in one plane) to a reference plane is to cause a force of magnitude equal to the centrifugal force of the revolving mass to act in the reference plane, together with a couple of magnitude equal to the product of the force and the distance between the plane of rotation and the reference plane. In order to have a complete balance of the several revolving masses in different planes, the following two conditions must be satisfied :

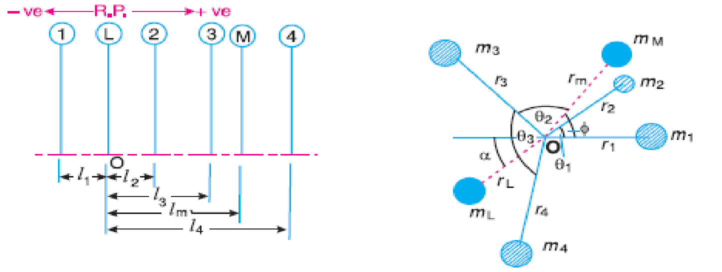
The relative angular positions of these masses are shown in the end view . The magnitude of the balancing masses m_L and m_M in planes *L* and *M* may be obtained as discussed below :

1. Take one of the planes, say *L* as the reference plane (*R.P.*). The distances of all the other planes to the left of the reference plane may be regarded as ***negative,*** and those to the right as ***positive.***

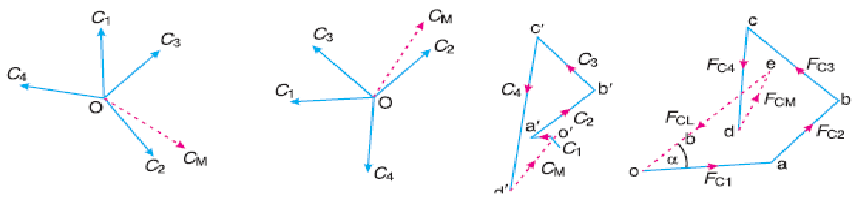
2.Tabulate the data as shown in fig The planes are tabulated in the same order in which they occur, reading from left to right.

Table 21.1

Plane (1)	Mass (m) (2)	Radius(r) (3)	Cent.force $\div \omega^2$ (m.r) (4)	Distance from Plane L (l) (5)	Couple $\div \omega^2$ (m.r.l) (6)
1	m_1	r_1	$m_1.r_1$	$-l_1$	$-m_1.r_1.l_1$
L(R.P.)	m_L	r_L	$m_L.r_L$	0	0
2	m_2	r_2	$m_2.r_2$	l_2	$m_2.r_2.l_2$
3	m_3	r_3	$m_3.r_3$	l_3	$m_3.r_3.l_3$
M	m_M	r_M	$m_M.r_M$	l_M	$m_M.r_M.l_M$
4	m_4	r_4	$m_4.r_4$	l_4	$m_4.r_4.l_4$



(*a*) Position of planes of the masses. (*b*) Angular position of the masses.



3. A couple may be represented by a vector drawn perpendicular to the plane of the couple. The couple C_1 introduced by transferring m_1 to the reference plane through O is proportional to $m_1.r_1.l_1$ and acts in a plane through Om_1 and perpendicular to the paper. The vector representing this couple is drawn in the plane of the paper and perpendicular to Om_1 as shown by OC_1 in Fig. Similarly, the vectors OC_2, OC_3 and OC_4 are drawn perpendicular to Om_2, Om_3 and Om_4 respectively and in the plane of the paper.

4. The couple vectors as discussed above, are turned counter clockwise through a right angle for convenience of drawing as shown in Fig. We see that their relative positions remains unaffected. Now the vectors OC_2, OC_3 and OC_4 are parallel and in the same direction as Om_2, Om_3 and Om_4, while the vector OC_1 is parallel to Om_1 but in *opposite direction. Hence the ***couple vectors are drawn radially outwards for the masses on one side of the reference plane and radially inward for the masses on the other side of the reference plane.***

5. Now draw the couple polygon as shown in Fig. . The vector $d'o'$ represents the balanced couple. Since the balanced couple C_M is proportional to $m_M.r_M.l_M$, therefore

$$C_M = m_M \cdot r_M \cdot l_M = \text{vector } d'o' \quad \text{or} \quad m_M = \frac{\text{vector } d'o'}{r_M \cdot l_M}$$

From this expression, the value of the balancing mass m_M in the plane M may be obtained, and the angle of inclination φ of this mass may be measured from Fig.

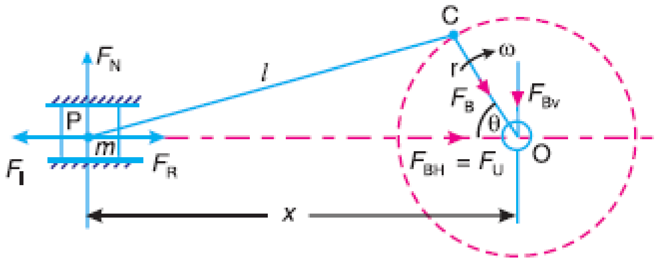
6. Now draw the force polygon as shown in Fig. The vector eo (in the direction from e to o) represents the balanced force. Since the balanced force is proportional to $m_L.r_L$, therefore,

$$m_L \cdot r_L = \text{vector } eo \quad \text{or} \quad m_L = \frac{\text{vector } eo}{r_L}$$

From this expression, the value of the balancing mass m_L in the plane L may be obtained and the angle of inclination α of this mass with the horizontal may be measured from Fig.

Balancing of Reciprocating Masses

The various forces acting on the reciprocating parts of an engine. The resultant of all the forces acting on the body of the engine due to inertia forces only is known as ***unbalanced force*** or ***shaking force***. Thus if the resultant of all the forces due to inertia effects is zero, then there will be no unbalanced force, but even then an unbalanced couple or shaking couple will be present Consider a horizontal reciprocating engine mechanism as shown in Fig.



F_R = Force required to accelerate the reciprocating parts,

F_B = Force acting on the crankshaft bearing or main bearing

Since F_R and F_I are equal in magnitude but opposite in direction, therefore they balance each other. The horizontal component of F_B (*i.e.* F_{BH}) acting along the line of reciprocation is also equal and opposite to F_I. This force $F_{BH} = F_U$ is an unbalanced force or shaking force and required to be properly balance.

The force on the sides of the cylinder walls (F_N) and the vertical component of F_B (*i.e.* F_{BV}) are equal and opposite and thus form a shaking couple of magnitude $F_N \times x$ or $F_{BV} \times x$.

From above we see that the effect of the reciprocating parts is to produce a shaking force and a shaking couple. Since the shaking force and a shaking couple vary in magnitude and direction during the engine cycle, therefore they cause very objectionable vibrations.

Thus the purpose of balancing the reciprocating masses is to eliminate the shaking force and a shaking couple. In most of the mechanisms, we can reduce the shaking force and a shaking couple by adding appropriate balancing mass, but it is usually not practical to eliminate them completely. In other words, the reciprocating masses are only partially balanced.

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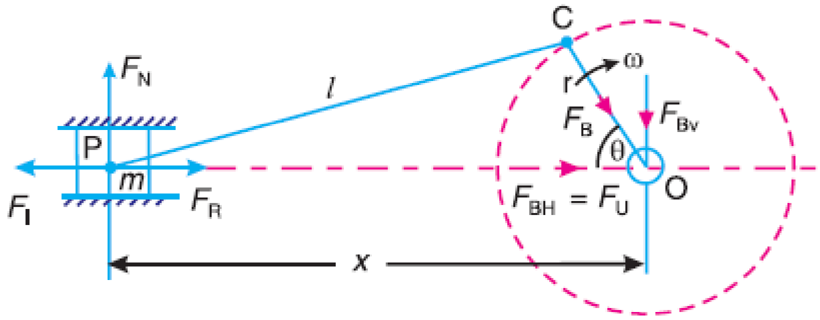

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Primary and Secondary Unbalanced Forces of Reciprocating Masses

Consider a reciprocating engine mechanism as shown in Fig.

Let m = Mass of the reciprocating parts,

l = Length of the connecting rod PC,

r = Radius of the crank OC,

θ = Angle of inclination of the crank with the line of stroke PO,

ω = Angular speed of the crank,

n = Ratio of length of the connecting rod to the crank radius = l / r.

$$a_R = \omega^2 \cdot r\left(\cos\theta + \frac{\cos 2\theta}{n}\right)$$

Inertia force due to reciprocating parts or force required to accelerate the reciprocating parts.

$$F_I = F_R = \text{Mass} \times \text{acceleration} = m \cdot \omega^2 \cdot r\left(\cos\theta + \frac{\cos 2\theta}{n}\right)$$

We have discussed in the previous article that the horizontal component of the force exerted on the crank shaft bearing (*i.e.* *F*BH) is equal and opposite to inertia force (*F*I). This force is an unbalanced one and is denoted by *F*U.

□Unbalanced force,

$$F_U = m \cdot \omega^2 \cdot r\left(\cos\theta + \frac{\cos 2\theta}{n}\right) = m.\omega^2 \cdot r\cos\theta + m \cdot \omega^2 \cdot r \times \frac{\cos 2\theta}{n} = F_P + F_S$$

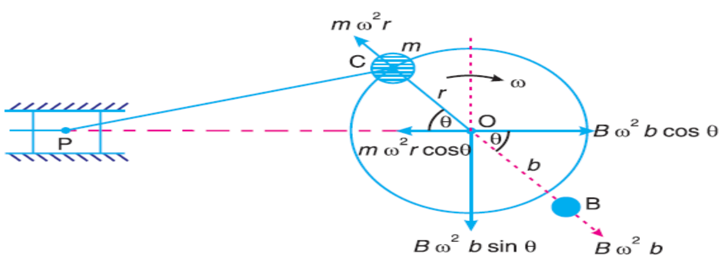
The expression $(m \cdot \omega^2 \cdot r\cos\theta)$ is known as ***primary unbalanced force*** and $\left(m \cdot \omega^2 \cdot r \times \frac{\cos 2\theta}{n}\right)$ is called ***secondary unbalanced force***.

$\therefore$ Primary unbalanced force, $F_P = m \cdot \omega^2 \cdot r\cos\theta$

and secondary unbalanced force, $F_S = m \cdot \omega^2 \cdot r \times \frac{\cos 2\theta}{n}$

Partial Balancing of Unbalanced Primary Force in a Reciprocating Engine

The primary unbalanced force $(m \cdot \omega^2 \cdot r\cos\theta)$ may be considered as the component of the centrifugal force produced by a rotating mass *m* placed at the crank radius *r*, as shown in Fig.



The primary force acts from *O* to *P* along the line of stroke. Hence, balancing of primary force is considered as equivalent to the balancing of mass *m* rotating at the crank radius *r*. This is balanced by having a mass *B* at a radius *b*, placed diametrically opposite to the crank pin *C*.

We know that centrifugal force due to mass *B*,

$$= B \cdot \omega^2 \cdot b$$

horizontal component of this force acting in opposite direction of primary force

$$= B \cdot \omega^2 \cdot b \cos\theta$$

The primary force is balanced, if

$$B \cdot \omega^2 \cdot b \cos\theta = m \cdot \omega^2 \cdot r \cos\theta \quad \text{or} \quad B.b = m.r$$

A little consideration will show, that the primary force is completely balanced if B.b = m.r, but the centrifugal force produced due to the revolving mass B, has also a vertical component (perpendicular to the line of stroke) of magnitude $B \cdot \omega^2 \cdot b \sin\theta$. This force remains unbalanced

The maximum value of this force is equal to $B \cdot \omega^2 \cdot b$ when θ is 90° and 270°, which is same as the maximum value of the primary force $m \cdot \omega^2 \cdot r$

let a fraction '*c*' of the reciprocating masses is balanced, such that

$$c.m.r = B.b$$

∴ Unbalanced force along the line of stroke

$$= m \cdot \omega^2 \cdot r \cos\theta - B \cdot \omega^2 \cdot b \cos\theta$$

$$= m \cdot \omega^2 \cdot r \cos\theta - c \cdot m \cdot \omega^2 \cdot r \cos\theta \qquad \ldots (\because B.b = c.m.r)$$

$$= (1 - c) m \cdot \omega^2 \cdot r \cos\theta$$

and unbalanced force along the perpendicular to the line of stroke

$$= B \cdot \omega^2 \cdot b \sin\theta = c \cdot m \cdot \omega^2 \cdot r \sin\theta$$

∴ Resultant unbalanced force at any instant

$$= \sqrt{\left[(1-c) m \cdot \omega^2 \cdot r \cos\theta\right]^2 + \left[c \cdot m \cdot \omega^2 \cdot r \sin\theta\right]^2}$$

$$= m \cdot \omega^2 \cdot r \sqrt{(1-c)^2 \cos^2\theta + c^2 \sin^2\theta}$$

Partial Balancing of Locomotives

The locomotives, usually, have two cylinders with cranks placed at right angles to each other in order to have uniformity in turning moment diagram. The two cylinder locomotives maybe classified as:

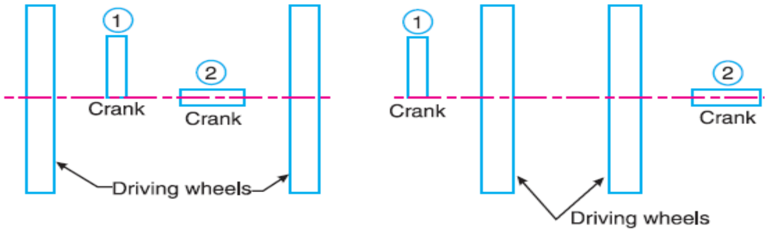
1. Inside cylinder locomotives

2. Outside cylinder locomotives.

In the ***inside cylinder locomotives***, the two cylinders are placed in between the planes of two driving wheels as shown in Fig(*a*) ; whereas in the ***outside cylinder locomotives***, the two cylinders are placed outside the driving wheels, one on each side of the driving wheel, as shown in
Fig(*b*). The locomotives may be

(*a*) Single or uncoupled locomotives

(*b*) Coupled locomotives



(*a*) Inside cylinder locomotives. (*b*) Outside cylinder locomotives.

A ***single*** or ***uncoupled locomotive*** is one, in which the effort is transmitted to one pair of the wheels only; whereas in ***coupled locomotives,*** the driving wheels are connected to the leading and trailing wheel by an outside coupling rod.

Effect of Partial Balancing of Reciprocating Parts of Two Cylinder Locomotives

We have discussed in the previous article that the reciprocating parts are only partially balanced. Due to this partial balancing of the reciprocating parts, there is an unbalanced primary force along the line of stroke and also an unbalanced primary force perpendicular to the line of stroke. The effect of an unbalanced primary force along the line of stroke is to produce;

1. Variation interactive force along the line of stroke
2. Swaying couple.

The effect of an unbalanced primary force perpendicular to the line of stroke is to produce variation in pressure on the rails, which results in hammering action on the rails. The maximum magnitude of the unbalanced force along the perpendicular to the line of stroke is known as a ***hammer blow***. We shall now discuss the effects of an unbalanced primary force in the following articles.

Variation of Tractive Force

The resultant unbalanced force due to the two cylinders, along the line of stroke, is known as ***tractive force***. Let the crank for the first cylinder be inclined at an angle θ with the line of stroke, as shown in Fig. Since the crank for the second cylinder is at right angle to the first crank, therefore the angle of inclination for the second crank will be (90° +θ).

Let m = Mass of the reciprocating parts per cylinder, and
c = Fraction of the reciprocating parts to be balanced.

We know that unbalanced force along the line of stroke for cylinder 1

$$=2\,(1-c)m.\omega^2\,.r\cos\theta$$

Similarly, unbalanced force along the line of stroke for cylinder 2,

$$=(1-c)m.\omega^2\,r\cos(90°+\theta)$$

As per definition, the tractive force,

Definition, the tractive force,

FT = Resultant unbalanced force along the line of stroke

$$=(1-c)m.\omega^2\,.r\cos\theta+(1-c)m.\omega^2\,r\cos(90°+\theta)$$

$$(1-c)m.\omega^2\,.r(\cos\theta-\sin\theta)$$

The tractive force is maximum or minimum when (cos θ – sin θ) is maximum or minimum. For (cosθ – sinθ) to be maximum or minimum,

$$\frac{d}{d\theta}(\cos\theta-\sin\theta)=0 \quad \text{or} \quad -\sin\theta-\cos\theta=0 \quad \text{or} \quad -\sin\theta=\cos\theta$$

$$\therefore \quad \tan\theta=-1 \quad \text{or} \quad \theta=135° \quad \text{or} \quad 315°$$

Thus, the tractive force is maximum or minimum when θ = 135° or 315°.

Maximum and minimum value of the tractive force or the variation in tractive force

$$=\pm(1-c)m.\omega^2.r(\cos 135°-\sin 135°)=\pm\sqrt{2}\,(1-c)\,m.\omega^2.r$$

PROBLEMS

Example1. The three cranks of a three cylinder locomotive are all on the same axle and are set at 120°. The pitch of the cylinders is 1 metre and the stroke of each piston is 0.6 m. The reciprocating masses are 300 kg for inside cylinder and 260 kg for each outside cylinder and the planes of rotation of the balance masses are 0.8 m from the inside crank.

If 40% of the reciprocating parts are to be balanced, find :

1. the magnitude and the position of the balancing masses required at a radius of 0.6 m ;

2. the hammer blow per wheel when the axle makes 6 r.p.s.

Solution. $\angle AOB = \angle BOC = \angle COA = 120°$; $l_A = l_B = l_C = 0.6$ m or $r_A = r_B = r_C$ =0.3 m ; $m_I = 300$ kg ; $m_O = 260$ kg ; $c = 40\% = 0.4$; $b_1 = b_2 = 0.6$ m ; $N = 6$ r.p.s.= $6 \times 2\pi$= 37.7 rad/s

Since 40% of the reciprocating masses are to be balanced, therefore mass of the reciprocating parts to be balanced for each outside cylinder,

$m_A = m_C = c \times m_O = 0.4 \times 260 = 104$ kg

and mass of the reciprocating parts to be balanced for inside cylinder,

$m_B = c \times m_1 = 0.4 \times 300 = 120$ kg

1. *Magnitude and position of the balancing masses*

B_1 and B_2 = Magnitude of the balancing masses in kg,

θ_1 and θ_2 = Angular position of the balancing masses B_1 and B_2 from crank *A*.

The magnitude and position of the balancing masses may be determined graphically as discussed below :

1. First of all, draw the position of planes and cranks as shown in Fig(*a*) and (*b*)respectively. The position of crank *A* is assumed in the horizontal direction.
2. Tabulate the data as given in the following table. Assume the plane of balancing mass B_1(*i.e.* plane 1) as the reference plane.

Plane (1)	*Mass* (m)kg (2)	*Radius* (r) m (3)	*Cent. force* $\div\omega^2$ (m.r) kg-m (4)	*Distance from plane1* (l)m (5)	*Couple* $\div\omega^2$ (m.r.l.) kg-m^2 (6)
A	104	0.3	31.2	– 0.2	– 6.24
1 (*R.P.*)	B_1	0.6	0.6 B_1	0	0
B	120	0.3	36	0.8	28.8
2	B_2	0.6	0.6 B_2	1.6	0.96 B_2
C	104	0.3	31.2	1.8	56.16

3. Now draw the couple polygon with the data given in Table (column 6), to some suitable scale, as shown in Fig.(*c*). The closing side *c' o'* represents the balancing couple and it is proportional to 0.96 B_2. Therefore, by measurement,
   $0.96\, B_2$ = vector $c'o'$ = 55.2 kg-m^2 or $B_2 = 57.5$ kg

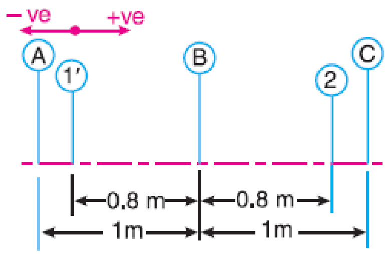
4. To determine the angular position of the balancing mass B_2, draw OB_2 parallel to vector*c' o* 'as shown in Fig (*b*). By measurement,

$\theta_2 = 24°$

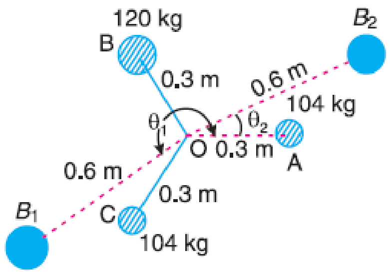
4. In order to find the balance mass B_1, draw the force polygon with the data given in Table (column 4), to some suitable scale, as shown in Fig(*d*). The closing side *co* represents the balancing force and it is proportional to 0.6 B_1.Therefore, by measurement,0.6 B_1 = vector *co* = 34.5 kg-m or B_1 = 57.5 kg

6. To determine the angular position of the balancing mass B_1, draw OB_1 parallel to vector *co*, as shown in Fig(*b*). By measurement,

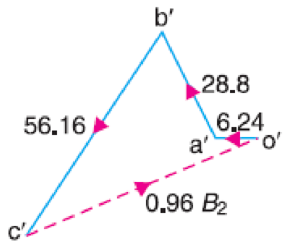
$$\theta_1 = 215°$$



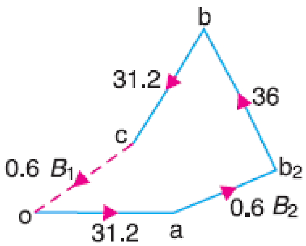
(*a*) Position of planes.



(*b*) Position of cranks.



(*c*) Couple polygon.



(*d*) Force polygon.

2.Hammer blow per wheel

We know that hammer blow per wheel
$=B_1\omega^2 b_1= 57.5\ (37.7)^2\ 20.6 = 49\ 035N$

Balancing of Coupled Locomotives

The uncoupled locomotives as discussed in the previous article, are obsolete now a days. In a coupled locomotive, the driving wheels are connected to the leading and trailing wheels by an outside coupling rod. By such an arrangement, a greater portion of the engine mass is utilized by tractive purposes. In coupled locomotives, the coupling rod cranks are placed diametrically opposite to the adjacent main cranks (*i.e.* driving cranks). The coupling rods together with cranks and pins may be treated as rotating masses and completely balanced by masses in the respective wheels. Thus in a coupled engine, the rotating and reciprocating masses must be treated separately and the balanced masses for the two systems are suitably combined in the wheel.

It may be noted that the variation of pressure between the wheel and the rail (*i.e.* hammer blow) may be reduced by equal distribution of balanced mass (*B*) between the driving, leading and trailing wheels respectively.

Balancing of Primary Forces of Multi-cylinder In-line Engines

The multi-cylinder engines with the cylinder centre lines in the same plane and on the same side of the centre line of the crankshaft are known as **In-line engines.** The following two conditions must be satisfied in order to give the primary balance of the reciprocating parts of a multi cylinder engine:

1. The algebraic sum of the primary forces must be equal to zero. In other words, the primary force polygon must *close

2. The algebraic sum of the couples about any point in the plane of the primary forces must be equal to zero. In other words, the primary couple polygon must close.

We have already discussed, that the primary unbalanced force due to the reciprocating masses is equal to the component, parallel to the line

of stroke, of the centrifugal force produced by the equal mass placed at the crankpin and revolving with it. Therefore, in order to give the primary balance of the reciprocating parts of a multi-cylinder engine, it is convenient to imagine the reciprocating masses to be transferred to their respective crank pins and to treat the problem as one of revolving masses.

Balancing of Secondary Forces of Multi-cylinder In-line Engines

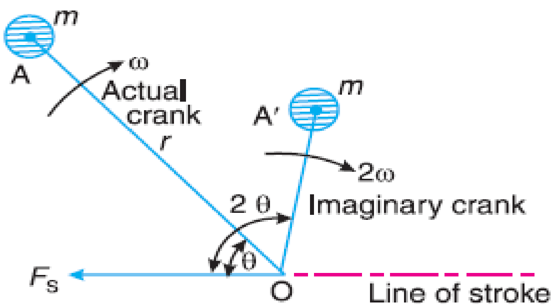
When the connecting rod is not too long (*i.e.* when the obliquity of the connecting rod is considered), then the secondary disturbing force due to the reciprocating mass arises.

$$F_S = m.\omega^2.r \times \frac{\cos 2\theta}{n}$$

This expression may be written as

$$F_S = m.(2\omega)^2 \times \frac{r}{4n} \times \cos 2\theta$$

As in case of primary forces, the secondary forces may be considered to be equivalent to the component, parallel to the line of stroke, of the centrifugal force produced by an equal mass placed at the imaginary crank of length $r / 4n$ and revolving at twice the speed of the actual crank (*i.e.* 2ω) as shown in Fig.



Thus, in multi-cylinder in-line engines, each imaginary secondary crank with a mass attached to the crankpin is inclined to the line of stroke at twice the angle of the actual crank. The values of the secondary forces and couples may be obtained by considering the revolving mass. This is done in the similar way as discussed for primary forces. The following two conditions must be satisfied in order to give a complete secondary balance of an engine:

1. The algebraic sum of the secondary forces must be equal to zero. In other words, the secondary force polygon must close, and

2. The algebraic sum of the couples about any point in the plane of the secondary forces must be equal to zero. In other words, the secondary couple polygon must close.

Example1. A four cylinder vertical engine has cranks 150 mm long. The planes of rotation of the first, second and fourth cranks are 400 mm, 200 mm and 200 mm respectively from the third crank and their reciprocating masses are 50 kg, 60 kg and 50 kg respectively. Find the mass of the reciprocating parts for the third cylinder and the relative angular positions of the cranks in order that the engine may be in complete primary balance.

Solution. Given $r_1 = r_2 = r_3 = r_4 = 150$ mm = 0.15 m ; $m_1 = 50$ kg ; $m_2 = 60$ kg ;
m4 = 50 kg

in order to give the primary balance of the reciprocating parts of a multi-cylinder engine, the problem may be treated as that of revolving masses with the reciprocating masses transferred to their respective crank pins.

The position of planes is shown in Fig(*a*). Assuming the plane of third cylinder as the reference plane, the data may be tabulated as given in Table.

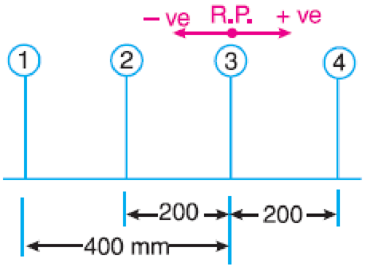
Plane (1)	*Mass (m) kg* (2)	*Radius (r) m* (3)	*Cent. force* $\div \omega^2$ *(m.r) kg-m* (4)	*Distance from plane 3(l) m* (5)	*Couple* $\div \omega^2$ *(m.r.l) kg-m*^{2} (6)
1	50	0.15	7.5	– 0.4	– 3
2	60	0.15	9	– 0.2	– 1.8
3(*R.P.*)	m_3	0.15	$0.15m_3$	0	0
4	50	0.15	7.5	0.2	1.5

First of all, the angular position of cranks 2 and 4 are obtained by drawing the couple polygon from the data given in Table (column 6). Assume the position of crank 1 in the horizontal direction as shown in Fig (*b*),

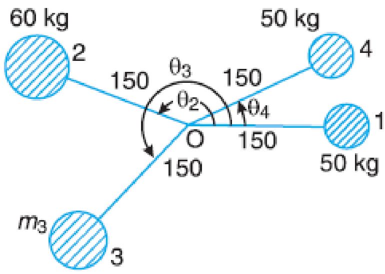
The couple polygon, as shown in Fig(*c*), is drawn as discussed below:

1. Draw vector o' a' in the horizontal direction (i.e. parallel to O_1) and equal to – 3 kg-m^2, to some suitable scale.

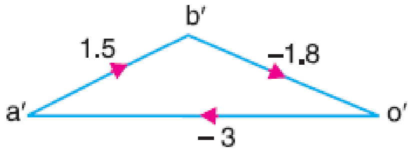
2. From point o'and a', draw vectors o' b' and a'b' equal to – 1.8 kg-m2 and 1.5 kg-m^{2}respectively. These vectors intersect at b'.



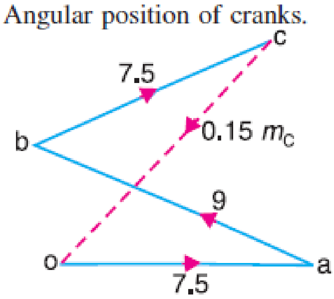
(*a*) Position of planes.



(*b*) Angular position of cranks.



(*c*) Couple polygon.



(*d*) Force polygon.

3. Now in Fig. 22.17 (*b*), draw *O*2 parallel to vector *o′ b′* and O_4 parallel to vector *a′ b′*. By measurement, we find that the angular position of crank 2 from crank 1 in the anticlockwise direction is

$$\theta_2 = 160°$$

and the angular position of crank 4 from crank 1 in the anticlockwise direction is

$$\theta 4 = 26°$$

In order to find the mass of the third cylinder (m_3) and its angular position, draw the force polygon, to some suitable scale, as shown in Fig (*d*), from the data given in Table(column 4). Since the closing side of the force polygon (vector *co*) is proportional to 0.15 m_3, therefore by measurement,

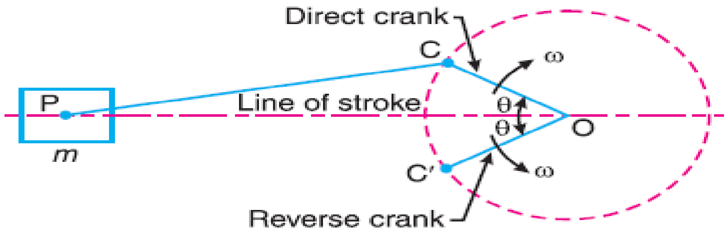
$$0.15 m_3 = 9 \text{ kg-m or } m_3 = 60 \text{ kg}$$

Now draw *O*3 in Fig 22.17 (*b*), parallel to vector *co*. By measurement, we find that the angular position of crank 3 from crank 1 in the anticlockwise direction is

$$\theta_3 = 227°$$

Balancing of Radial Engines (Direct and Reverse Cranks Method)

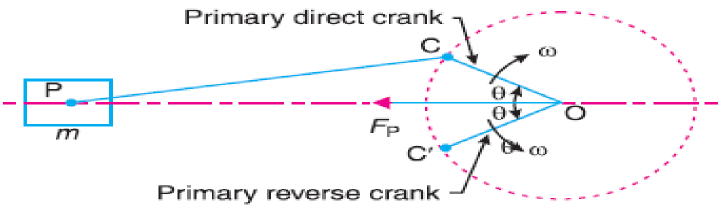
The method of direct and reverse cranks is used in balancing of radial or *V*-engines, in which the connecting rods are connected to a common crank. Since the plane of rotation of the various cranks (in radial or *V*-engines) is same, therefore there is no unbalanced primary or secondary couple.



Consider a reciprocating engine mechanism as shown in Fig. 22.27. Let the crank *OC* (known as the direct crank) rotates uniformly at ω radians per second in a clockwise direction. Let at any instant the crank makes an angle θ with the line of stroke *OP*. The indirect or reverse crank *OC′* is the image of the direct crank *OC*, when seen through the mirror placed at the line of stroke .A little consideration will show that when the direct crank revolves in a clockwise direction, the reverse crank will revolve in the anticlockwise direction. We shall now discuss the primary and secondary forces due to the mass (*m*) of the reciprocating parts at *P*.

Considering the primary forces

We have already discussed that primary force is $2\ m.\omega^2\ .r \cos\theta$. This force is equal to the component of the centrifugal force along the line of stroke, produced by a mass (*m*) placed at the crank pin *C*. Now let us suppose that the mass (*m*) of the reciprocating parts is divided into two parts, each equal to *m* / 2.



It is assumed that *m* / 2 is fixed at the ***direct crank*** (termed as ***primary direct crank***) pin *C* and *m* / 2 at the ***reverse crank*** (termed as ***primary reverse crank***) pin *C′*, as shown in Fig. We know that the centrifugal force acting on the primary direct and reverse crank

$$= \frac{m}{2} \times \omega^2 .r$$

∴ Component of the centrifugal force acting on the primary direct crank

$$= \frac{m}{2} \times \omega^2 .r \cos\theta \qquad \text{. . . (in the direction from } O \text{ to } P)$$

and, the component of the centrifugal force acting on the primary reverse crank

$$= \frac{m}{2} \times \omega^2 .r \cos\theta \qquad \text{. . . (in the direction from } O \text{ to } P)$$

∴ Total component of the centrifugal force along the line of stroke

$$= 2 \times \frac{m}{2} \times \omega^2 .r \cos\theta = m.\omega^2 .r \cos\theta = \text{Primary force, } F_P$$

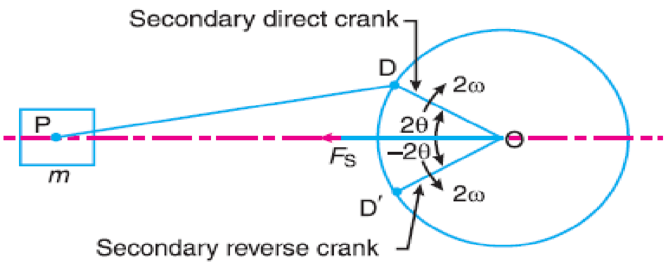
Hence, for primary effects the mass m of the reciprocating parts at P may be replaced by two masses at C and C′ each of magnitude m/2.

Considering secondary forces

We know that the secondary force

$$= m(2\omega)^2 \frac{r}{4n} \times \cos 2\theta = m.\omega^2 r. \times \frac{\cos 2\theta}{n}$$

In the similar way as discussed above, it will be seen that for the secondary effects, the mass (m) of the reciprocating parts may be replaced by two masses (each m/2) placed at D and D′ such that OD = OD′ = r/4n. The crank OD is the secondary direct crank and rotates at 2π rad/s in the clockwise direction, while the crank OD′ is the secondary reverse crank and rotates at 2πrad/s in the anticlockwise direction as shown in Fig.

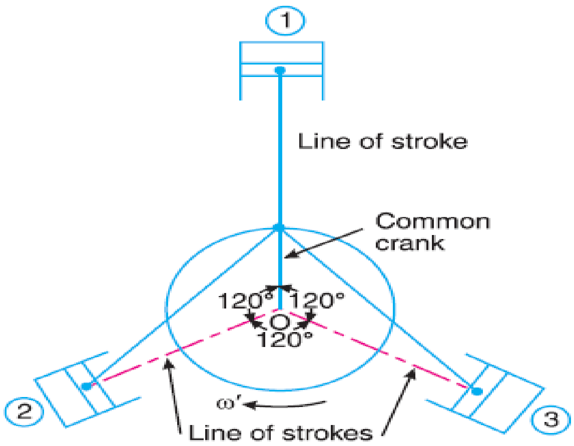


PROBLEMS

Example 1. The three cylinders of an air compressor have their axes 120° to one another, and their connecting rods are coupled to a single crank. The stroke is 100 mm and the length of each connecting rod is 150 mm. The mass of the reciprocating parts per cylinder is 1.5kg. Find the maximum primary and secondary forces acting on the frame of the compressor when running at 3000 r.p.m. Describe clearly a method by which such forces may be balanced.

Solution. Given : L = 100 mm or $r = L / 2$ = 50 mm = 0.05 m ; l = 150 mm = 0.15 m ;m = 1.5 kg ; N = 3000 r.p.m. or $\omega = 2\pi \times 3000/60 = 314.2$ rad/s

The position of three cylinders is shown in Fig. Let the common crank be along the inner dead centre of cylinder 1. Since common crank rotates clockwise, therefore θ is positive when measured clockwise.

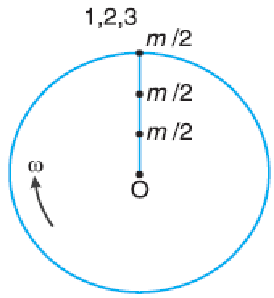


Maximum primary force acting on the frame of the compressor

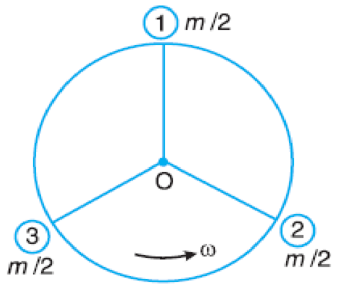
The primary direct and reverse crank positions as shown in Fig.(*a*) and (*b*), are obtained as discussed below :

1. Since θ = 0° for cylinder 1, therefore both the primary direct and reverse cranks will coincide with the common crank.
2. Since θ = ±120° for cylinder 2, therefore the primary direct crank is 120° clockwise and the primary reverse crank is 120° anti-clockwise
from the line of stroke of cylinder 2.
3. Since θ = ± 240° for cylinder 3, therefore the primary direct crank is 240° clockwise and the primary reverse crank is 240° anti-clockwise from the line of stroke of cylinder 3.From Fig.(*b*), we see that the primary reverse cranks form a balanced system. Therefore there is no unbalanced primary force due to the reverse cranks. From Fig (*a*), we see that the resultant primary force is equivalent to the centrifugal force of a mass 3 *m*/2 attached to the end of the crank.

$$\therefore \text{ Maximum primary force } = \frac{3m}{2} \times \omega^2 . r = \frac{3 \times 1.5}{2} (314.2)^2 0.05 = 11\,106 \text{ N} = 11.106 \text{ kN}$$



(*a*) Direct primary cranks.



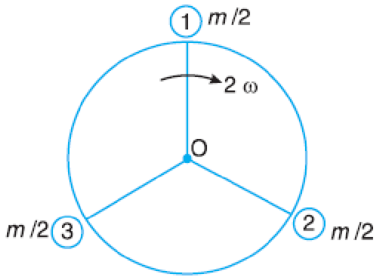
(*b*) Reverse primary cranks.

maximum primary force may be balanced by a mass attached diametrically opposite to the crank pin and rotating with the crank, of magnitude B_1 at radius b_1 such that

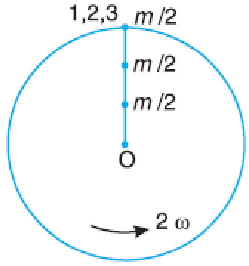
$$B_1 . b_1 = \frac{3m}{2} \times r = \frac{3 \times 1.5}{2} \times 0.05 = 0.1125 \text{ N-m}$$

Maximum secondary force acting on the frame of the compressor

The secondary direct and reverse crank positions as shown in Fig (*a*) and (*b*), are obtained as discussed below:
1. Since θ = 0° and 2 θ = 0° for cylinder 1, therefore both the secondary direct and reverse cranks will coincide with the common crank.
2. Since θ = ±120° and 2 θ = ± 240° for cylinder 2, therefore the secondary direct crank is 240° clockwise and the secondary reverse crank is 240° anticlockwise from the line of stroke of cylinder 2.
3. Since θ = ± 240° and 2 θ = ± 480°, therefore the secondary direct crank is 480° or 120°clockwise and the secondary reverse crank is 480° or 120° anti-clockwise from the line of stroke of cylinder 3.



(*a*) Direct secondary cranks.



(*b*) Reverse secondary cranks.

From Fig (*a*), we see that the secondary direct cranks form a balanced system. Therefore there is no unbalanced secondary force due to the direct cranks. From Fig (*b*), we see that the resultant secondary force is equivalent to the centrifugal force of a mass 3 *m*/2attached at a crank radius of *r*/4*n* and rotating at a speed of 2ωrad/s in the opposite direction to the crank.

∴ Maximum secondary force

$$= \frac{2m}{2}(2\omega)^2\left(\frac{r}{4n}\right) = \frac{3\times1.5}{2}(2\times314.2)^2\left[\frac{0.05}{4\times0.15/0.05}\right]\text{N}$$

$$\ldots(\because n = l/r)$$

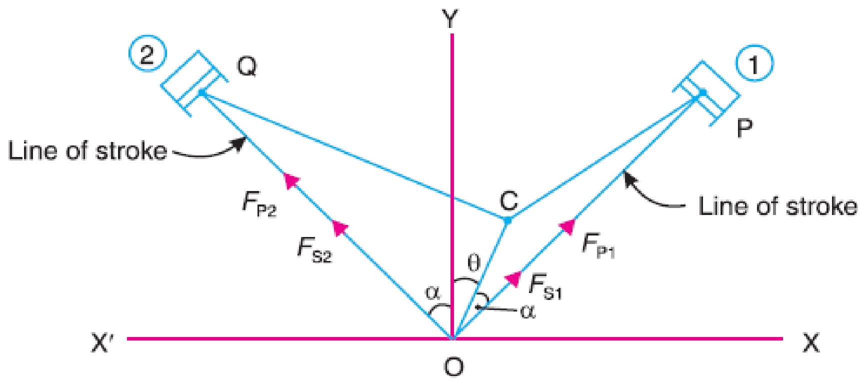
= 3702 N **Ans.**

This maximum secondary force may be balanced by a mass B_2 at radius b_2, attached diametrically opposite to the crankpin, and rotating anti-clockwise at twice the crank speed, such that

$$B_2.b_2 = \frac{3m}{2}\times\frac{r}{4n} = \frac{3\times1.5}{2}\times\frac{0.05}{4\times0.15/0.05} = 0.009\ 375\ \text{N-m}$$

Balancing of V-engines

Consider a symmetrical two cylinder *V*-engine as shown in Fig. 22.33,
The common crank *OC* is driven by two connecting rods *PC* and *QC*. The lines of stroke *OP* and *OQ* are inclined to the vertical *OY*, at an angle α as shown in Fig.



We know that inertia force due to reciprocating parts of cylinder 1, along the line of stroke

$$= m.\omega^2.r\left[\cos(\alpha-\theta)+\frac{\cos 2(\alpha-\theta)}{n}\right]$$

the inertia force due to reciprocating parts of cylinder 2, along the line of stroke

$$= m.\omega^2.r\left[\cos(\alpha-\theta)+\frac{\cos 2(\alpha+\theta)}{n}\right]$$

The balancing of *V*-engines is only considered for primary and secondary forces* as discussed below :

Considering primary forces

We know that primary force acting along the line of stroke of cylinder 1,

$F_{P1} = m.\omega^2.r\cos(\alpha-\theta)$

Component of F_{P1} along the vertical line OY

$= F_{P1}\cos\alpha = m.\omega^2 r.\cos(\alpha-\theta)\cos\alpha$ . . . . . .**(i)**

and component of F_{P1} along the horizontal line *OX*

$= F_{P1}\sin\alpha = m.\omega^2 r.\sin(\alpha-\theta)\sin\alpha$ . . . . . .**(ii)**

Similarly, primary force acting along the line of stroke of cylinder 2,

$F_{P2} = m.\omega^2.r\cos(\alpha+\theta)$

Component of F_{P2} along the vertical line *OY*

$F_{P2}=\cos\alpha == m.\omega^2 r.\cos(\alpha+\theta)\cos\alpha$**.........(iii)**

and component of F_{P2} along the horizontal line *OX′*

$= F_{P2}\sin\alpha = m.\omega^2 r.\sin(\alpha+\theta)\sin\alpha$ **........(iv)**

Total component of primary force along the vertical line *OY*

$$F_{PV} = (i) + (iii) = m.\omega^2.r\cos\alpha\,[\cos(\alpha-\theta)+\cos(\alpha+\theta)]$$

$$= m.\omega^2.r\cos\alpha\times 2\cos\alpha\cos\theta$$

$$\ldots\ [\because\ \cos(\alpha-\theta)+\cos(\alpha+\theta) = 2\cos\alpha\cos\theta]$$

$$= 2m.\omega^2.r\cos^2\alpha.\cos\theta$$

total component of primary force along the horizontal line *OX*

$$F_{PH} = (ii) - (iv) = m.\omega^2.r\sin\alpha[\cos(\alpha-\theta)-\cos(\alpha+\theta)]$$

$$= m.\omega^2.r\sin\alpha\times 2\sin\alpha\sin\theta$$

$$\ldots\ [\because\ \cos(\alpha-\theta)-\cos(\alpha+\theta) = 2\sin\alpha\sin\theta]$$

$$= 2m.\omega^2.r\sin^2\alpha.\sin\theta$$

∴ Resultant primary force,

$$F_P = \sqrt{(F_{PV})^2+(F_{PH})^2}$$

$$= 2m.\omega^2.r\sqrt{(\cos^2\alpha.\cos\theta)^2+(\sin^2\alpha.\sin\theta)^2} \quad \ldots (v)$$

Considering secondary forces

We know that secondary force acting along the line of stroke of cylinder 1,

$$F_{S1} = m.\omega^2.r \times \frac{\cos 2(\alpha - \theta)}{n}$$

Component of F_{S1} along the vertical line OY

$$= F_{S1} \cos\alpha = m.\omega^2.r \times \frac{\cos 2(\alpha - \theta)}{n} \times \cos\alpha \qquad \ldots (ix)$$

Component of F_{S1} along the horizontal line OX

$$= F_{S1} \sin\alpha = m.\omega^2.r \times \frac{\cos 2(\alpha - \theta)}{n} \times \sin\alpha \qquad \ldots (x)$$

Similarly, secondary force acting along the line of stroke of cylinder 2,

$$F_{S2} = m.\omega^2 r \times \frac{\cos 2(\alpha + \theta)}{n}$$

Component of F_{S2} along the vertical line OY

$$= F_{S2} \cos\alpha = m.\omega^2.r \times \frac{\cos 2(\alpha + \theta)}{n} \times \cos\alpha \qquad \ldots (xi)$$

Component of F_{S2} along the horizontal line OX'

$$= F_{S2} \sin\alpha = m.\omega^2.r \times \frac{\cos 2(\alpha + \theta)}{n} \times \sin\alpha \qquad \ldots (xii)$$

Total component of secondary force along the vertical line OY,

$$F_{SV} = (ix) + (xi) = \frac{m}{n} \times \omega^2.r \cos\alpha\,[\cos 2(\alpha - \theta) + \cos 2(\alpha + \theta)]$$

$$= \frac{m}{n} \times \omega^2.r \cos\alpha \times 2\cos 2\alpha \cos 2\theta = \frac{2m}{n} \times \omega^2.r \cos\alpha.\cos 2\alpha \cos 2\theta$$

total component of secondary force along the horizontal line OX,

$$F_{SH} = (x) - (xii) = \frac{m}{n} \times \omega^2.r \sin\alpha\,[\cos 2(\alpha - \theta) - \cos 2(\alpha + \theta)]$$

$$= \frac{m}{n} \times \omega^2.r \sin\alpha \times 2\sin 2\alpha.\sin 2\theta$$

$$= \frac{2m}{n} \times \omega^2.r \sin\alpha.\sin 2\alpha.\sin 2\theta$$

Resultant secondary force,

$$F_S = \sqrt{(F_{SV})^2 + (F_{SH})^2}$$

$$= \frac{2m}{n} \times \omega^2.r \sqrt{(\cos\alpha.\cos 2\alpha.\cos 2\theta)^2 + (\sin\alpha.\sin 2\alpha.\sin 2\theta)^2}$$

PROBLEMS

Example 1. A vee-twin engine has the cylinder axes at right angles and the connecting rods operate a common crank. The reciprocating mass per cylinder is 11.5 kg and the crank radius is 75 mm. The length of the connecting rod is 0.3 m. Show that the engine may be balanced for primary forces by means of a revolving balance mass.
If the engine speed is 500 r.p.m. What is the value of maximum resultant secondary force ?

Solution. Given : 2θ= 90° or α= 45° ; m = 11.5 kg ; r = 75 mm = 0.075 m ; l = 0.3 m ;N = 500 r.p.m. or ω= 2π×500 / 60 = 52.37 rad/s

We know that resultant primary force,

$$F_P = 2m.\omega^2.r\sqrt{(\cos^2\alpha\cos\theta)^2 + (\sin^2\alpha\sin\theta)^2}$$

$$= 2m.\omega^2.r\sqrt{(\cos^2 45°\cos\theta)^2 + (\sin^2 45°\sin\theta)^2}$$

$$= 2m.\omega^2.r\sqrt{\left[\frac{\cos\theta}{2}\right]^2 + \left[\frac{\sin\theta}{2}\right]^2} = m.\omega^2.r$$

Since the resultant primary force m.ω^2 .r is the centrifugal force of a mass m at the crank radius r when rotating at ω rad / s, therefore, the engine may be balanced by a rotating balance mass.

Maximum resultant secondary force

We know that resultant secondary force,

$$F_S = \sqrt{2}\times\frac{m}{n}\times\omega^2.r\sin 2\theta \qquad \text{. . . (When } 2\alpha = 90°)$$

This is maximum, when sin 2 θ is maximum *i.e.* when sin 2 θ = ± 1 or θ = 45° or 135°.
Maximum resultant secondary force,

$$F_{S_{max}} = \sqrt{2}\times\frac{m}{n}\times\omega^2.r \qquad \text{. . . (Substituting } \theta = 45°)$$

$$= \sqrt{2}\times\frac{11.5}{0.3/0.075}(52.37)^2 0.075 = 836 \text{ N } \textbf{Ans.} \qquad \text{. . . } (\because n = l/r)$$

VIBRATIONS

When elastic bodies such as a spring, a beam and a shaft are displaced from the equilibrium position by the application of external forces, and then released, they execute a vibratory motion. This is due to the reason that, when a body is displaced, the internal forces in the form of elastic or strain energy are present in the body. At release, these forces bring the body to its original position. When the body reaches the equilibrium position, the whole of the elastic or strain energy is converted into kinetic energy due to which the body continues to move in the opposite direction. The whole of the kinetic energy is again converted into strain energy due to which the body again returns to the equilibrium position. In this way, the vibratory motion is repeated indefinitely.

Terms Used in Vibratory Motion

The following terms are commonly used in connection with the vibratory motions :

1. ***Period of vibration or time period.*** It is the time interval after which the motion is repeated itself. The period of vibration is usually expressed in seconds.

2. ***Cycle.*** It is the motion completed during one time period.

3. ***Frequency.*** It is the number of cycles described in one second. In S.I. units, the frequency is expressed in hertz (briefly written as Hz) which is equal to one cycle per second.

Types of Vibratory Motion

The following types of vibratory motion are important from the subject point of view :

1. ***Free or natural vibrations.*** When no external force acts on the body, after giving it an initial displacement, then the body is said to be under ***free or natural vibrations.*** The frequency of the free vibrations is called ***free or natural frequency.***

2. ***Forced vibrations.*** When the body vibrates under the influence of external force, then the body is said to be under ***forced vibrations.*** The external force applied to the body is a periodic disturbing force created by unbalance. The vibrations have the same frequency as the applied force.

Note : When the frequency of the external force is same as that of the natural vibrations, resonance takes place.

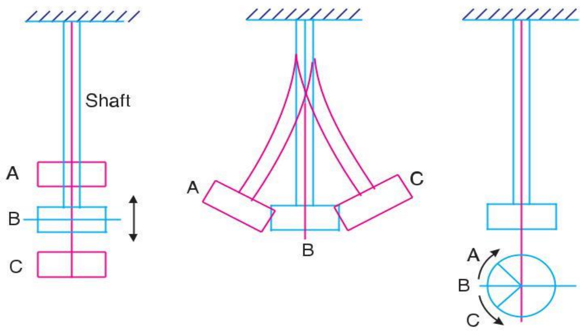
Damped vibrations. When there is a reduction in amplitude over every cycle of vibration, the motion is said to be ***damped vibration.*** This is due to the fact that a certain amount of energy possessed by the vibrating system is always dissipated in overcoming frictional resistances to the motion.

Types of Free Vibrations

The following three types of free vibrations are important from the subject point of view :

1. Longitudinal vibrations, **2.** Transverse vibrations, and **3.** Torsional vibrations.

Consider a weightless constraint (spring or shaft) whose one end is fixed and the other end carrying a heavy disc, as shown in Fig. This system may execute one of the three above mentioned types of vibrations



B = Mean position ; *A* and *C* = Extreme positions.

(*a*) Longitudinal vibrations. (*b*) Transverse vibrations. (*c*) Torsional vibrations.

1. ***Longitudinal vibrations*.** When the particles of the shaft or disc moves parallel to the axis of the shaft, as shown in Fig. (*a*), then the vibrations are known as ***longitudinal vibrations*.** In this case, the shaft is elongated and shortened alternately and thus the tensile and compressive stresses are induced alternately in the shaft.

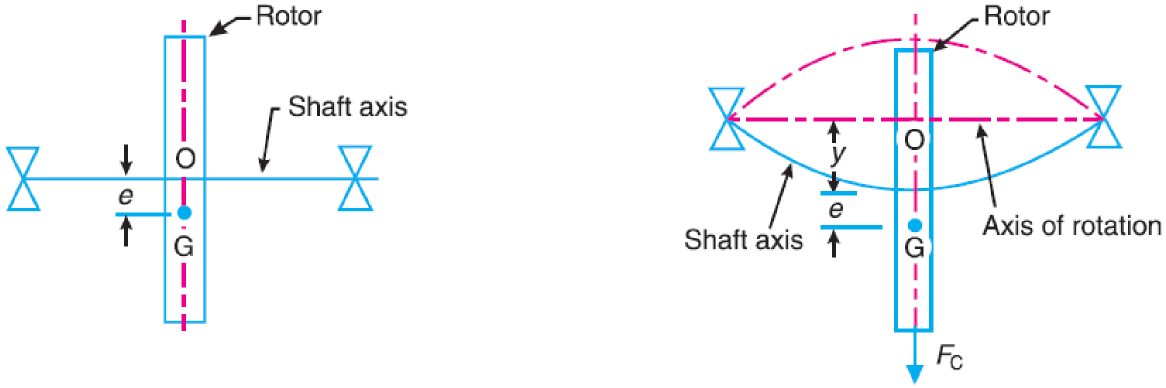
***2 . Transverse vibrations*.** When the particles of the shaft or disc move approximately perpendicular to the axis of the shaft, as shown in Fig (*b*), then the vibrations are known as ***transverse vibrations*.** In this case, the shaft is straight and bent alternately and bending stresses are induced in the shaft

3.*Torsional vibrations***. When the particles of the shaft or disc move in a circle about the axis of the shaft, as shown in Fig. (*c*), then the vibrations are known as ***torsional vibrations*.** In this case, the shaft is twisted and untwisted alternately and the torsional shear stresses are induced in the shaft.

Critical or Whirling Speed of a Shaft

A rotating shaft carries different mountings and accessories in the form of gears, pulleys, etc. When the gears or pulleys are put on the shaft, the centre of gravity of the Pulley or gear does not coincide with the centre line of the bearings or with the axis of the shaft, when the shaft is stationary. This means that the centre of gravity of the pulley or gear is at a certain distance from the axis of rotation and due to this, the shaft is subjected to centrifugal force.

This force will bent the shaft which will further increase the distance of centre of gravity of the pulley or gear from the axis of rotation. This correspondingly increases the value of centrifugal force, which further increases the distance of centre of gravity from the axis of rotation. This effect is cumulative and ultimately the shaft fails. The bending of shaft not only depends upon the value of eccentricity (distance between centre of gravity of the pulley and the axis of rotation) but also depends upon the speed at which the shaft rotates **The speed at which the shaft runs so that the additional deflection of the shaft from the axis of rotation becomes infinite, is known as *critical* or *whirling speed*.**



(*a*) When shaft is stationary. (*b*) When shaft is rotating.

Consider a shaft of negligible mass carrying a rotor, as shown in Fig.(*a*). The point *O* is on the shaft axis and *G* is the centre of gravity of the rotor. When the shaft is stationary, the centre line of the bearing and the axis of the shaft coincides. Fig.(*b*) shows the shaft when rotating about the axis of rotation at a uniform speed of ω rad/s.

Let *m* = Mass of the rotor,
e = Initial distance of centre of gravity of the rotor from the centre line of the bearing or shaft axis, when the shaft is stationary,

y = Additional deflection of centre of gravity of the rotor when the shaft starts rotating at ω rad/s, and
s = Stiffness of the shaft *i.e.* the load required per unit deflection of the shaft.
Since the shaft is rotating at ω rad/s, therefore centrifugal force acting radially outwards through *G* causing the shaft to deflect is given by The shaft behaves like a spring.

$$F_C = m.\omega^2 (y + e)$$

Therefore the force resisting the deflection *y*, =*s.y*
For the equilibrium position,

$$m.\omega^2 (y + e) = s.y$$

$$m.\omega^2.y + m.\omega^2.e = s.y \quad \text{or} \quad y(s - m.\omega^2) = m.\omega^2.e$$

$$y = \frac{m.\omega^2.e}{s - m.\omega^2} = \frac{\omega^2.e}{s/m - \omega^2}$$

We know that circular frequency,

$$\omega_n = \sqrt{\frac{s}{m}} \qquad \text{or} \qquad y = \frac{\omega^2 . e}{(\omega_n)^2 - \omega^2}$$

A little consideration will show that when $\omega > \omega_n$, the value of y will be negative and the shaft deflects is the opposite direction as shown dotted in Fig

$$y = \pm \frac{\omega^2 e}{(\omega_n)^2 - \omega^2} = \frac{\pm e}{\left(\frac{\omega_n}{\omega}\right)^2 - 1} = \frac{\pm e}{\left(\frac{\omega_c}{\omega}\right)^2 - 1}$$

We see from the above expression that when $\omega_n = \omega_c$ the value of y becomes infinite. Therefore ω_c is the **critical or whirling speed.**

$$\omega_c = \omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{g}{\delta}} \text{ Hz}$$

If N_c is the critical or whirling speed in r.p.s., then

$$2\pi N_c = \sqrt{\frac{g}{\delta}} \qquad \text{or} \qquad N_c = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}} = \frac{0.4985}{\sqrt{\delta}} \text{ r.p.s.}$$

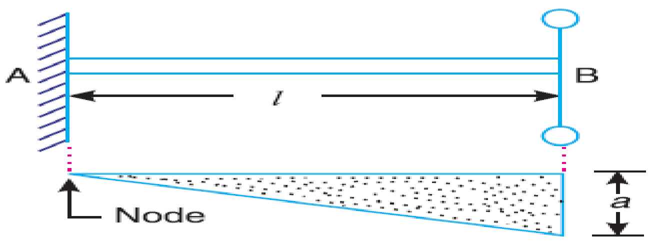
δ = Static deflection of the shaft in metres.

Hence the **critical or whirling speed is the same as the natural frequency of transverse vibration but its unit will be revolutions per second.**

Free Torsional Vibrations of a Single Rotor System

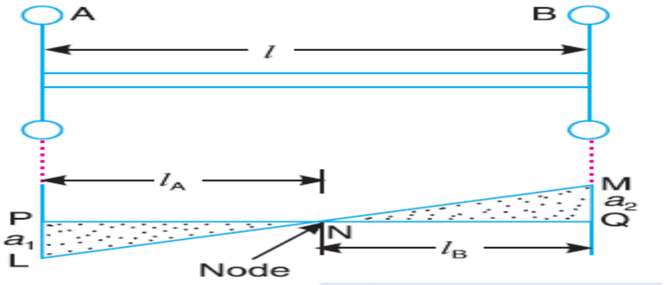
A shaft is fixed at one end and carrying a rotor at the free end.The natural frequency of torsional vibration

$$f_n \quad \frac{1}{2}\sqrt{\frac{q}{I}} \quad \frac{1}{2}\sqrt{\frac{C.J}{l.I}}$$



Free Torsional Vibrations of a Two Rotor System

Consider two rotor systems. It consists of a shaft with two rotors at its ends. In this system torsional vibrations occurs only when the two rotors A and B move in opposite directions .It may be noted that the two rotors must have same frequency. The node lies at Point N. This point can be safely assumed as fixed end of the shaft may be considered as two separate shafts NP and NQ each fixed to one of its ends and carrying rotors at the free ends



Let

l=Length of the shaft
l_A=Length of the part NP'
l_B=Length of the part NQ
I_A=Mass moment of inertia of rotor A
I_B=Mass moment of inertia of rotor B
d= diameter of the shaft
J=polar moment of inertia of shaft
C = Modulus of rigidity for shaft
then natural frequency for Rotor A

$$f_{nA} \quad \frac{1}{2} \sqrt{\frac{C.J}{l_A.I_A}}$$

natural frequency for Rotor B

$$f_{nB} \quad \frac{1}{2} \sqrt{\frac{C.J}{l_B.I_B}}$$

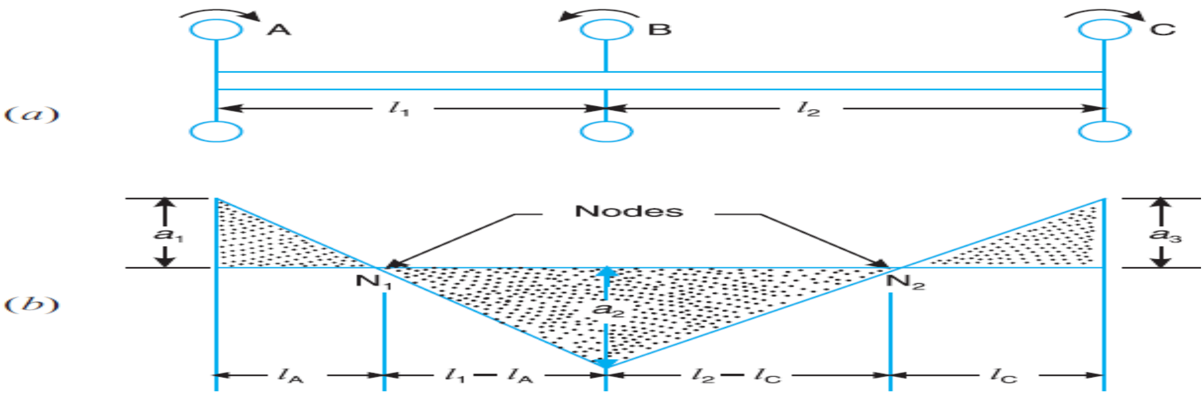
Since $f_{nA} = f_{nB}$, therefore

$$\frac{1}{2} \sqrt{\frac{C.J}{l_A.I_A}} \quad \frac{1}{2} \sqrt{\frac{C.J}{l_B.I_B}} \quad \text{or} \quad l_A . I_A = l_B . I_B$$

$$l_A \quad \frac{l_B.I_B}{I_A}$$

Free Torsional Vibrations of a Three Rotor System

Consider three rotors. It consists of a shaft and three rotors A,B and C. The rotors A and C are attached to the ends of a shaft where as the rotor B is attached in between A and C .The torsional vibrations may occur in two ways that is with either one node or two nodes



Natural frequency for rotor A is

$$f_{nA} \quad \frac{1}{2}\sqrt{\frac{C.J}{l_A.I_A}}$$

Natural frequency for rotor B is

$$*f_{nB} \quad \frac{1}{2}\sqrt{\frac{C.J}{I_B}\quad\frac{1}{l_1 \quad l_A}\quad\frac{1}{l_2 \quad l_C}}$$

Natural frequency for rotor C is

$$f_{nC} \quad \frac{1}{2}\sqrt{\frac{C.J}{l_C.I_C}}$$

Since $f_{nA} = f_{nB} = f_{nC},$

$$\frac{1}{2}\sqrt{\frac{C.J}{l_A.I_A}} \quad \frac{1}{2}\sqrt{\frac{C.J}{l_C.I_C}} \qquad \text{or} \qquad l_A.I_A = l_C.I_C$$

$$l_A \quad \frac{l_C.I_C}{I_A}$$

Now equating equations

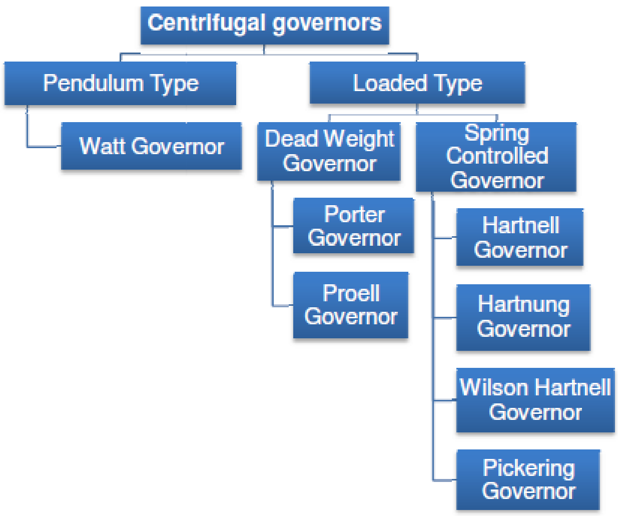
$$\frac{1}{2}\sqrt{\frac{C.J}{I_B}\quad\frac{1}{l_1 \quad l_A}\quad\frac{1}{l_2 \quad l_C}} \quad \frac{1}{2}\sqrt{\frac{C.J}{l_C.I_C}}$$

UNIT-V
GOVERNORS

INTRODUCTION

The function of a governor is to regulate the mean speed of an engine, when there are variations in the load *e.g.* when the load on an engine increases, its speed decreases, Therefore it becomes necessary to increase the supply of working fluid. On the other hand, when the load on the engine decreases, its speed increases and thus less working fluid is required. The governor automatically controls the supply of working fluid to the engine with the varying load condition sand keeps the mean speed within certain limits. A little consideration will show, that when the load increases, the configuration of the governor changes and a valve is moved to increase the supply of the working fluid ;***conversely,*** when the load decreases, the engine speed increases and the governor decreases the supply of working fluid.

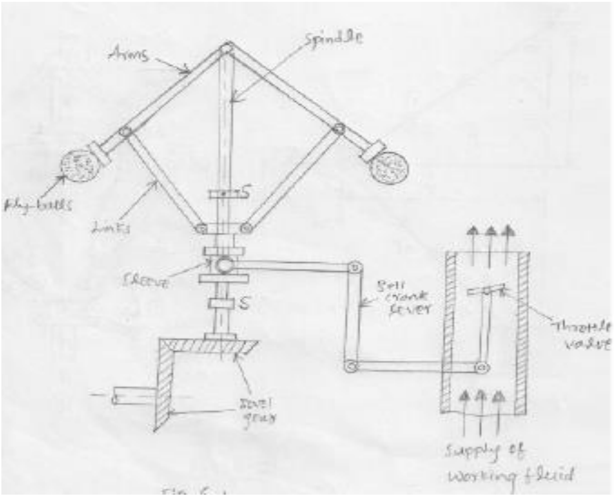
Classifications of the governor



Centrifugal Governors

The centrifugal governors are based on the balancing of centrifugal force on the rotating balls by an equal and opposite radial force, known as the controlling force*.It consists of two balls of equal mass, which are attached to the arms as shown in Fig. These balls are known as governor balls or fly balls. The balls revolve with a spindle, which is driven by the engine through bevel gears. The upper ends of the arms are pivoted to the spindle, so that the balls may rise up or fall down as they revolve about the vertical axis. The arms are connected by the links to a sleeve, which is keyed to the spindle. This sleeve revolves with the spindle ; but can slide up and down. The balls and the sleeve rises when the spindle speed increases, and falls when the speed decreases. In order to limit the travel of the sleeve in upward and downward directions, two stops *S*, *S* are provided on the spindle. The sleeve is connected by a bell crank lever to a throttle valve. The supply of the working fluid decreases when the sleeve rises and increases when it falls.

When the load on the engine increases, the engine and the governor speed decreases. This results in the decrease of centrifugal force on the balls. Hence the balls move inwards and the sleeve moves downwards. The downward movement of the sleeve operates a throttle valve at the other end of the bell crank lever to increase the supply of working fluid and thus the engine speed is increased.



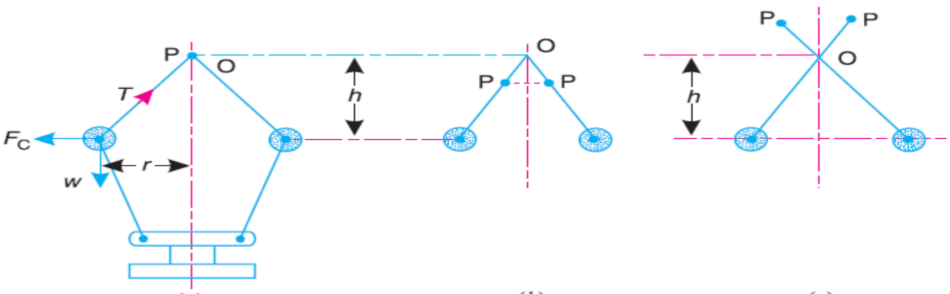
In this case, the extra power output is provided to balance the increased load. When the load on the engine decreases, the engine and the governor speed increases, which results in the increase of centrifugal force on the balls. Thus the balls move outwards and the sleeve rises upwards. This upward movement of the sleeve reduces the supply of the working fluid and hence the speed is decreased. In this case, the power output is reduced.

Watt Governor

The simplest form of a centrifugal governor is a Watt governor, as shown in Fig. It is basically a conical pendulum with links attached to a sleeve of negligible mass. The arms of the g overnor may be connected to the spindle in the following three ways

governor may be connected to the spindle in the following three ways :

1. The pivot *P*, may be on the spindle axis as shown in Fig. (*a*).
2. The pivot *P*, may be offset from the spindle axis and the arms when produced intersect at*O*, as shown in Fig.(*b*).
3. The pivot *P*, may be offset, but the arms cross the axis at *O*, as shown in Fig(a)



(*a*) (*b*) (*c*)

Let m = Mass of the ball in kg,

w = Weight of the ball in newtons = $m.g$,

T = Tension in the arm in newtons,

ω = Angular velocity of the arm and ball about the spindle axis in rad/s,

r = Radius of the path of rotation of the ball *i.e.* horizontal distance from the centre of the ball to the spindle axis in metres,

F_C = Centrifugal force acting on the ball in newtons = $m.\omega^2.r$, and

h = Height of the governor in metres.

It is assumed that the weight of the arms, links and the sleeve are negligible as compared to the weight of the balls. Now, the ball is in equilibrium under the action of

1. the centrifugal force (FC) acting on the ball,

2. the tension (T) in the arm, and

3. the weight(w) of the ball.

Taking moments about point O, we have

$F_C \times h = w \times r = m.g.r$ or

$m.\omega^2.r.h = m.g.r$ or $h = g/\omega^2$

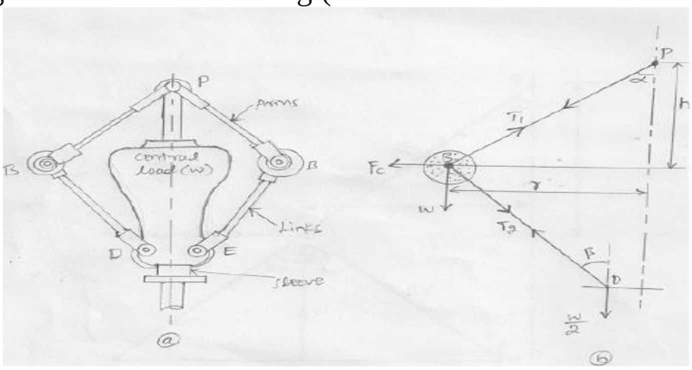
When g is expressed in m/s^2 and ω in rad/s, then h is in metres. If N is the speed in r.p.m., then

$\omega = 2\pi N/60$

$$h = \frac{9.81}{(2\pi N/60)^2} = \frac{895}{N^2} \text{ metres}$$

Porter Governor

The Porter governor is a modification of a Watt's governor, with central load attached to the sleeve as shown in Fig (a). The load moves up and down the central spindle. This additional downward force increases the speed of revolution required to enable the balls to rise to any predetermined level. Consider the forces acting on one-half of the governor as shown in Fig (b



Let m = Mass of each ball in kg,
w = Weight of each ball in newtons = $m.g$,
M = Mass of the central load in kg,
W = Weight of the central load in newtons = $M.g$,
r = Radius of rotation in metres,
h = Height of governor in metres ,
N = Speed of the balls in r.p.m .,
ω= Angular speed of the balls in rad/s
= $2\pi N/60$ rad/s,
F_C = Centrifugal force acting on the ball in newtons = $m.\omega^2.r$,
T_1 = Force in the arm in newtons,
T_2 = Force in the link in newtons,
α= Angle of inclination of the arm (or upper link) to the vertical, and
β= Angle of inclination of the link (or lower link) to the vertical.

Though there are several ways of determining the relation between the height of the governor (h) and the angular speed of the balls (ω), yet the following two methods are important from the subject point of view.

1. Method of resolution of forces
2. Instantaneous centre method

1.Method of resolution of forces

Considering the equilibrium of the forces acting at D, we have

$$T_2 \cos\beta = \frac{W}{2} = \frac{M.g}{2}$$

$$T_2 = \frac{M.g}{2\cos\beta}$$

Again, considering the equilibrium of the forces acting on B. The point B is in equilibrium under the action of the following forces, as shown in Fig(b).

(i) The weight of ball ($w = m.g$),
(ii) The centrifugal force (FC),
(iii) The tension in the arm (T1), and
(iv) The tension in the link (T2).

Resolving the forces vertically,

$$T_1 \cos\alpha = T_2 \cos\beta + w = \frac{M.g}{2} + m.g \qquad \ldots (ii)$$

Resolving the forces horizontally,

$$T_1 \sin\alpha + T_2 \sin\beta = F_C$$

$$T_1 \sin\alpha + \frac{M.g}{2\cos\beta} \times \sin\beta = F_C \qquad \ldots\left(\because T_2 = \frac{M.g}{2\cos\beta}\right)$$

$$T_1 \sin\alpha + \frac{M.g}{2} \times \tan\beta = F_C$$

$$T_1 \sin\alpha = F_C - \frac{M.g}{2} \times \tan\beta \qquad \ldots (iii)$$

Dividing equation ***(iii)*** by equation ***(ii)***,

$$\frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \frac{F_C - \dfrac{M.g}{2} \times \tan\beta}{\dfrac{M.g}{2} + m.g}$$

or
$$\left(\frac{M.g}{2} + m.g\right)\tan\alpha = F_C - \frac{M.g}{2} \times \tan\beta$$

$$\frac{M.g}{2} + m.g = \frac{F_C}{\tan\alpha} - \frac{M.g}{2} \times \frac{\tan\beta}{\tan\alpha}$$

Substituting $\dfrac{\tan\beta}{\tan\alpha} = q$, and $\tan\alpha = \dfrac{r}{h}$, we have

$$\frac{M.g}{2} + m.g = m.\omega^2.r \times \frac{h}{r} - \frac{M.g}{2} \times q \qquad \ldots (\because F_C = m.\omega^2.r)$$

$$m.\omega^2.h = m.g + \frac{M.g}{2}(1+q)$$

$$\therefore \qquad h = \left[m.g + \frac{M.g}{2}(1+q)\right]\frac{1}{m.\omega^2} = \frac{m + \dfrac{M}{2}(1+q)}{m} \times \frac{g}{\omega^2} \qquad \ldots (iv)$$

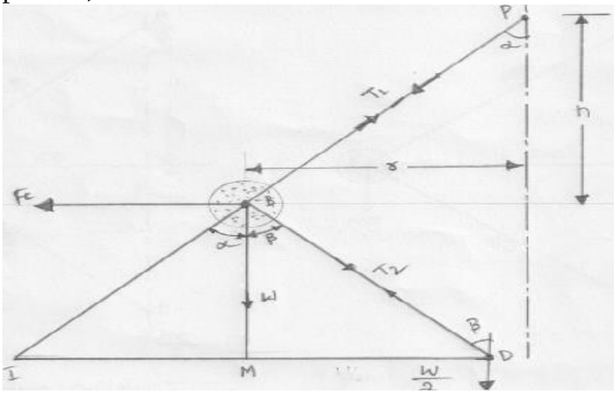
$$\omega^2 = \left[m.g + \frac{Mg}{2}(1+q)\right]\frac{1}{m.h} = \frac{m + \dfrac{M}{2}(1+q)}{m} \times \frac{g}{h}$$

$$\left(\frac{2\pi N}{60}\right)^2 = \frac{m + \dfrac{M}{2}(1+q)}{m} \times \frac{g}{h}$$

$$N^2 = \frac{m + \dfrac{M}{2}(1+q)}{m} \times \frac{g}{h}\left(\frac{60}{2\pi}\right)^2 = \frac{m + \dfrac{M}{2}(1+q)}{m} \times \frac{895}{h} \qquad \ldots (v)$$

2. Instantaneous centre method

In this method, equilibrium of the forces acting on the link *BD* are considered. The instantaneous centre *I* lies at the point of intersection of *PB* produced and a line through *D* perpendicular to the spindle axis, as shown in Fig. 18.4.Taking moments about the point *I*,



$$F_C \times BM = w \times IM + \frac{W}{2} \times ID$$

$$= m.g \times IM + \frac{M.g}{2} \times ID$$

$$F_C = m.g \times \frac{IM}{BM} + \frac{M.g}{2} \times \frac{ID}{BM}$$

$$= m.g \times \frac{IM}{BM} + \frac{M.g}{2}\left(\frac{IM + MD}{BM}\right)$$

$$= m.g \times \frac{IM}{BM} + \frac{M.g}{2}\left(\frac{IM}{BM} + \frac{MD}{BM}\right)$$

$$= m.g \tan\alpha + \frac{M.g}{2}(\tan\alpha + \tan\beta)$$

$$\therefore\ m.\omega^2.r \times \frac{h}{r} = m.g + \frac{M.g}{2}(1+q)$$

$$h = \frac{m.g + \frac{M.g}{2}(1+q)}{m} \times \frac{1}{\omega^2} = \frac{m + \frac{M}{2}(1+q)}{m} \times \frac{g}{\omega^2}$$

. . . (Same as before)

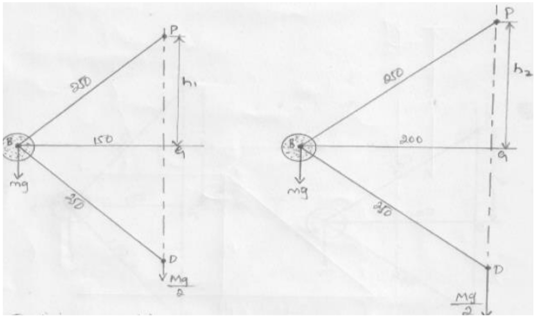
When $\tan\alpha = \tan\beta$ or $q = 1$, then

$$h = \frac{m+M}{m} \times \frac{g}{\omega^2}$$

PROBLEMS

Example1. A Porter governor has equal arms each 250 mm long and pivoted on the axis of rotation. Each ball has a mass of 5 kg and the mass of the central load on the sleeve is 25 kg. The radius of rotation of the ball is 150 mm when the governor begins to lift and 200 mm when the governor is at maximum speed. Find the minimum and maximum speeds and range of speed of the governor.

Solution. Given : $BP = BD$ = 250 mm = 0.25 m ; m = 5 kg ; M = 15 kg ;
r_1 = 150 mm= 0.15m; r_2 = 200 mm = 0.2 m



The minimum and maximum positions of the governor are shown in Fig.(a) and (b)respectively.

Minimum speed when r_1 = BG = 0.15 m

Let N_1 = Minimum speed.

From Fig.(a), we find that height of the governor,

$$h_1 = PG = \sqrt{(PB)^2 - (BG)^2} = \sqrt{(0.25)^2 - (0.15)^2} = 0.2 \text{ m}$$

We know that

$$(N_1)^2 = \frac{m + M}{m} \times \frac{895}{h_1} = \frac{5 + 15}{5} \times \frac{895}{0.2} = 17\ 900$$

$N1$ = 133.8 r.p.m.
Maximum speed when r_2 = BG = 0.2 m
Let $N2$ = Maximum speed.
From Fig(b), we find that height of the governor,

$$h_2 = PG = \sqrt{(PB)^2 - (BG)^2} = \sqrt{(0.25)^2 - (0.2)^2} = 0.15 \text{ m}$$

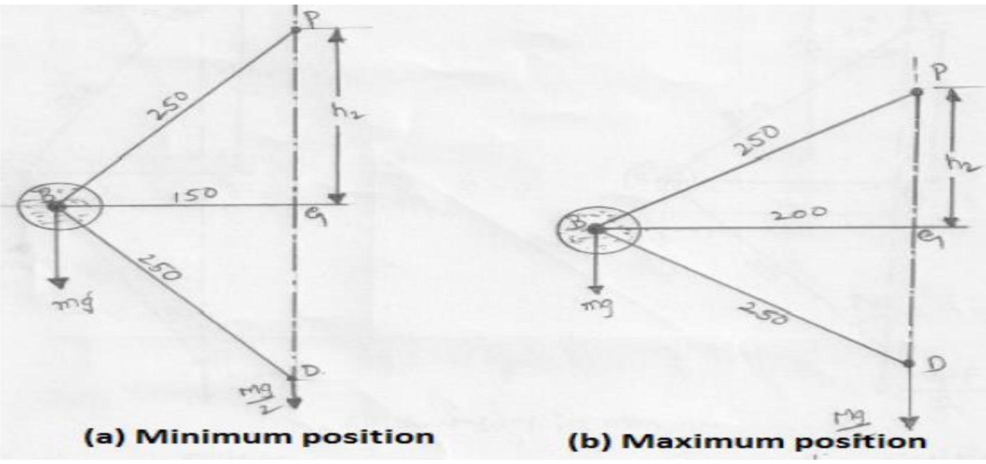
$$(N_2)^2 = \frac{m+M}{m} \times \frac{895}{h_2} = \frac{5+15}{5} \times \frac{895}{0.15} = 23\ 867$$

$N2$ = 154.5 r.p.m.
Range of speed

We know that range of speed= $N_2 - N_1$ = 154.4 – 133.8 = 20.7 r.p.m.

Example2. The arms of a Porter governor are each 250 mm long and pivoted on the governor axis. The mass of each ball is 5 kg and the mass of the central sleeve is 30 kg. The radius of rotation of the balls is 150 mm when the sleeve begins to rise and reaches a value of 200 mm for maximum speed. Determine the speed range of the governor. If the friction at the sleeve is equivalent of 20 N of load at the sleeve, determine how the speed range is modified.

Solution. Given : $BP = BD = 250$ mm ; $m = 5$ kg ; $M = 30$ kg ;
$r_1 = 150$ mm ; $r_2 = 200$ mm
First of all, let us find the minimum and maximum speed of the governor. The minimum and
maximum position of the governor is shown in Fig (*a*) and (*b*) respectively.
Let N_1 = Minimum speed when $r_1 = BG = 150$ mm, and
N_2 = Maximum speed when $r_2 = BG = 200$ mm.



Speed range of the governor

From Fig (*a*), we find that height of the governor,

$$h_1 = PG = \sqrt{(PB)^2 - (BG)^2} = \sqrt{(250)^2 - (150)^2} = 200 \text{ mm} = 0.2 \text{ m}$$

We know that

$$(N_1)^2 = \frac{m+M}{m} \times \frac{895}{h_1} = \frac{5+30}{5} \times \frac{895}{0.2} = 31\ 325$$

$$N_1 = 177 \text{ r.p.m.}$$

From Fig(*b*), we find that height of the governor,

$$h_2 = PG = \sqrt{(PB)^2 - (BG)^2} = \sqrt{(250)^2 - (200)^2} = 150 \text{ mm} = 0.15 \text{ m}$$

We know that

$$(N_2)^2 = \frac{m + M}{m} \times \frac{895}{h_2} = \frac{5 + 30}{5} \times \frac{895}{0.15} = 41\,767$$

$$N_2 = 204.4 \text{ r.p.m.}$$

We know that speed range of the governor

= $N2 - N1$ = 204.4 – 177 = 27.4 r.p.m

Speed range when friction at the sleeve is equivalent of 20 N of load (i.e. when F = 20 N**)**

We know that when the sleeve moves downwards, the friction force (*F*) acts upwards and the minimum speed is given by

$$(N_1)^2 = \frac{m.g + (M.g - F)}{m.g} \times \frac{895}{h_1}$$

$$= \frac{5 \times 9.81 + (30 \times 9.81 - 20)}{5 \times 9.81} \times \frac{895}{0.2} = 29500$$

$$\therefore \quad N_1 = 172 \text{ r.p.m.}$$

We also know that when the sleeve moves upwards, the frictional force (*F*) acts downwards and the maximum speed is given by

$$(N_2)^2 = \frac{m.g + (M.g + F)}{m.g} \times \frac{895}{h_2}$$

$$= \frac{5 \times 9.81 + (30 \times 9.81 + 20)}{5 \times 9.81} \times \frac{895}{0.15} = 44\,200$$

$$\therefore \quad N_2 = 210 \text{ r.p.m.}$$

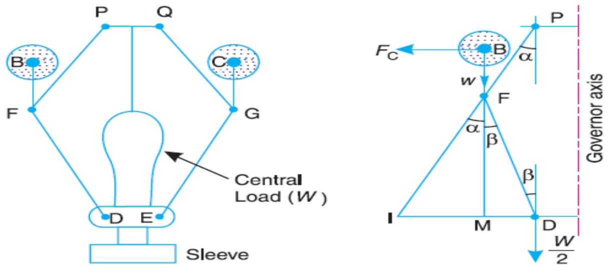
We know that speed range of the governor

= $N_2 - N_1$ = 210 – 172 = 38 r.p.m.

Proell Governor

The Proell governor has the balls fixed at *B* and *C* to the extension of the links *DF* and *EG*, as shown in Fig(*a*). The arms *FP* and *GQ* are pivoted at *P* and *Q* respectively.

Consider the equilibrium of the forces on one-half of the governor as shown in Fig (*b*).The instantaneous centre (*I*) lies on the intersection of the line *PF* produced and the line from *D* drawn perpendicular to the spindle axis. The perpendicular *BM* is drawn on *ID*.



Taking moments about I,

$$F_C \times BM = w \times IM + \frac{W}{2} \times ID = m.g \times IM + \frac{M.g}{2} \times ID \qquad \ldots (i)$$

$$\therefore \qquad F_C = m.g \times \frac{IM}{BM} + \frac{M.g}{2}\left(\frac{IM + MD}{BM}\right) \qquad \ldots (\because\ ID = IM + MD)$$

Multiplying and dividing by FM, we have

$$F_C = \frac{FM}{BM}\left[m.g \times \frac{IM}{FM} + \frac{M.g}{2}\left(\frac{IM}{FM} + \frac{MD}{FM}\right)\right]$$

$$= \frac{FM}{BM}\left[m.g \times \tan\alpha + \frac{M.g}{2}(\tan\alpha + \tan\beta)\right]$$

$$= \frac{FM}{BM} \times \tan\alpha\left[m.g + \frac{M.g}{2}\left(1 + \frac{\tan\beta}{\tan\alpha}\right)\right]$$

We know that $F_C = m.\omega^2 r$; $\tan\alpha = \frac{r}{h}$ and $q = \frac{\tan\beta}{\tan\alpha}$

$$\therefore \qquad m.\omega^2.r = \frac{FM}{BM} \times \frac{r}{h}\left[m.g + \frac{M.g}{2}(1 + q)\right]$$

and

$$\omega^2 = \frac{FM}{BM}\left[\frac{m + \frac{M}{2}(1 + q)}{m}\right]\frac{g}{h} \qquad \ldots (ii)$$

Substituting $\omega = 2\pi N/60$, and $g = 9.81$ m/s^2, we get

$$N^2 = \frac{FM}{BM}\left[\frac{m + \frac{M}{2}(1 + q)}{m}\right]\frac{895}{h} \qquad \ldots (iii)$$

PROBLEMS

Example 1. A Proell governor has equal arms of length 300 mm. The upper and lower ends of the arms are pivoted on the axis of the governor. The extension arms of the lower links are each 80 mm long and parallel to the axis when the radii of rotation of the balls are 150 mm and200 mm. The mass of each ball is 10 kg and the mass of the central load is 100 kg. Determine the range of speed of the governor.

Solution. Given : $PF = DF = 300$ mm ; $BF = 80$ mm ; $m = 10$ kg ; $M = 100$ kg ;
$r_1 = 150$ mm; $r_2 = 200$ mm

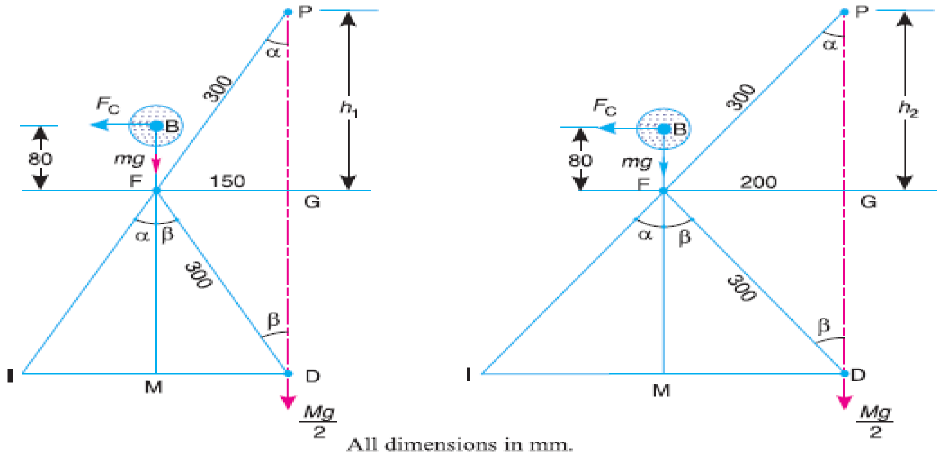
First of all, let us find the minimum and maximum speed of the governor. The minimum and
maximum position of the governor is shown in Fig (a)

Let

N_1 = Minimum speed when radius of rotation, $r_1 = FG = 150$ mm ;

N_2 = Maximum speed when radius of rotation , $r_2 = FG = 200$ mm.

From Fig(a), we find that height of the governor,



All dimensions in mm.

(a) Minimum position. (a) Maximum position.

$$h_1 = PG = \sqrt{(PF)^2 - (FG)^2} = \sqrt{(300)^2 - (150)^2} = 260 \text{ mm} = 0.26 \text{ m}$$

$FM = GD = PG = 260$ mm $= 0.26$ m

$BM = BF + FM = 80 + 260 = 340$ mm $= 0.34$ m

We know that $$(N_1)^2 = \frac{FM}{BM}\left(\frac{m+M}{m}\right)\frac{895}{h_1} \quad \ldots (\therefore \alpha = \beta \text{ or } q = 1)$$

$$= \frac{0.26}{0.34}\left(\frac{10+100}{10}\right)\frac{895}{0.26} = 28\,956 \quad \text{or} \quad N_1 = 170 \text{ r.p.m.}$$

Now from Fig. 18.13 (b), we find that height of the governor,

$$h_2 = PG = \sqrt{(PF)^2 - (FG)^2} = \sqrt{(300)^2 - (200)^2} = 224 \text{ mm} = 0.224 \text{ m}$$

$FM = GD = PG = 224$ mm $= 0.224$ m

$\therefore$ $BM = BF + FM = 80 + 224 = 304$ mm $= 0.304$ m

We know that $$(N_2)^2 = \frac{FM}{BM}\left(\frac{m+M}{m}\right)\frac{895}{h_2} \quad \ldots (\because \alpha = \beta \text{ or } q = 1)$$

$$= \frac{0.224}{0.304}\left(\frac{10+100}{10}\right)\frac{895}{0.224} = 32\,385 \quad \text{or} \quad N_2 = 180 \text{ r.p.m.}$$

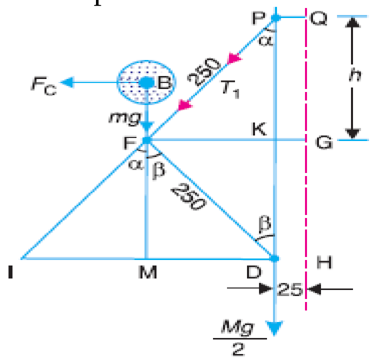
We know that range of speed

$= N_2 - N_1 = 180 - 170 = 10$ r.p.m. **Ans.**

Example2 . A governor of the Proell type has each arm 250 mm long. The pivots of the upper and lower arms are 25 mm from the axis. The central load acting on the sleeve has a mass of25 kg and the each rotating ball has a mass of 3.2 kg. When the governor sleeve is in mid-position, the extension link of the lower arm is vertical and the radius of the path of rotation of the masses is175 mm. The vertical height of the governor is 200 mm.

If the governor speed is 160 r.p.m. when in mid-position, find :
1. Length of the extension link;
2. Tension in the upper arm.

Solution. Given : $PF = DF = 250$ mm = 0.25 m ; $PQ = DH = KG = 25$ mm = 0.025 m ;M = 25 kg ; $m = 3.2$ kg ; $r = FG = 175$ mm = 0.175 m ; $h = QG = PK = 200$ mm = 0.2 m ; N = 160 r.p.m.



Let BF = Length of the extension link.
The Proell governor in its mid-position is shown in Fig.
From the figure, we find that
$FM = GH = QG = 200$ mm = 0.2 m
We know that

$$N^2 = \frac{FM}{BM}\left(\frac{m+M}{m}\right)\frac{895}{h}$$

. . . ($\because$ $\alpha = \beta$ or $q = 1$)

$$(160)^2 = \frac{0.2}{BM}\left(\frac{3.2+25}{3.2}\right)\frac{895}{0.2} = \frac{7887}{BM}$$

$$BM = 7887/(160)^2 = 0.308 \text{ m}$$

$BF = BM - FM = 0.308 - 0.2 = 0.108$ m = 108 mm

2. *Tension in the upper arm*

Let T_1 = Tension in the upper arm.

$$PK = \sqrt{(PF)^2 - (FK)^2} = \sqrt{(PF)^2 - (FG - KG)^2}$$

$$= \sqrt{(250)^2 - (175-25)^2} = 200 \text{ mm}$$

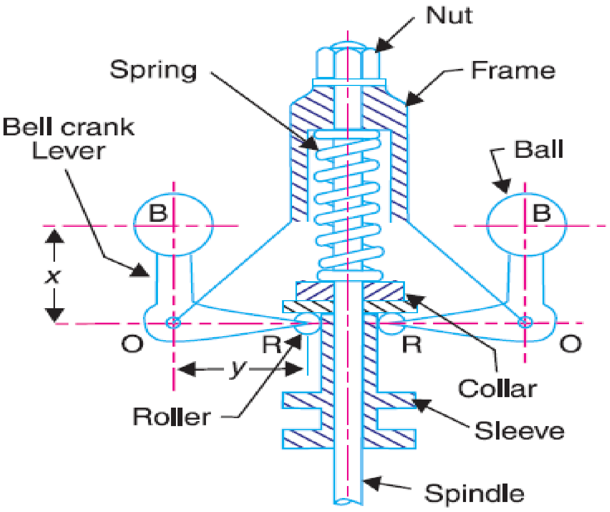
$$\cos\alpha = PK/PF = 200/250 = 0.8$$

$$T_1 \cos\alpha = mg + \frac{Mg}{2} = 3.2 \times 9.81 + \frac{25 \times 9.81}{2} = 154 \text{ N}$$

$$\therefore \quad T_1 = \frac{154}{\cos\alpha} = \frac{154}{0.8} = 192.5 \text{ N} \quad \textbf{Ans.}$$

Hartnell Governor

A Hartnell governor is a spring loaded governor as shown in Fig. It consists of two bell crank levers pivoted at the points *O*, *O* to the frame. The frame is attached to the governor spindle and therefore rotates with it. Each lever carries a ball at the end of the vertical arm *OB* and a roller at the end of the horizontal arm *OR*. A helical spring in compression provides equal downward forces on the two rollers through a collar on the sleeve. The spring force may be adjusted by screwing a nut up or down on the sleeve.



m = Mass of each ball in kg,
M = Mass of sleeve in kg,
r_1 = Minimum radius of rotation in metres,
r_2 = Maximum radius of rotation in metres,
ω_1 = Angular speed of the governor at minimum radius in rad/s,
ω_2 = Angular speed of the governor at maximum radius in rad/s,
S_1 = Spring force exerted on the sleeve at ω_1 in newtons,
S_2 = Spring force exerted on the sleeveat ω_2 in newtons,
F_{C1} = Centrifugal force at ω_1 in newtons = $m(\omega_1)^2 r_1$,
F_{C2} = Centrifugal force at ω_2 in newtons = $m(\omega_2)^2 r_2$,
s = Stiffness of the spring or the force required to compress the spring by one mm,
x = Length of the vertical or ball arm of the lever in metres,
y = Length of the horizontal or sleeve arm of the lever in metres, and
r = Distance of fulcrum *O* from the governor axis or the radius of rotation when thegovernor is in mid-position, in metres.

Consider the forces acting at one bell crank lever. The minimum and maximum position is shown in Fig. Let h be the compression of the spring when the radius of rotation changes from r_1 to r_2.

For the minimum position *i.e.* when the radius of rotation changes from r to r_1, as shown in Fig (*a*), the compression of the spring or the lift of sleeve h_1 is given by

$$\frac{h_1}{y} = \frac{a_1}{x} = \frac{r - r_1}{x}$$

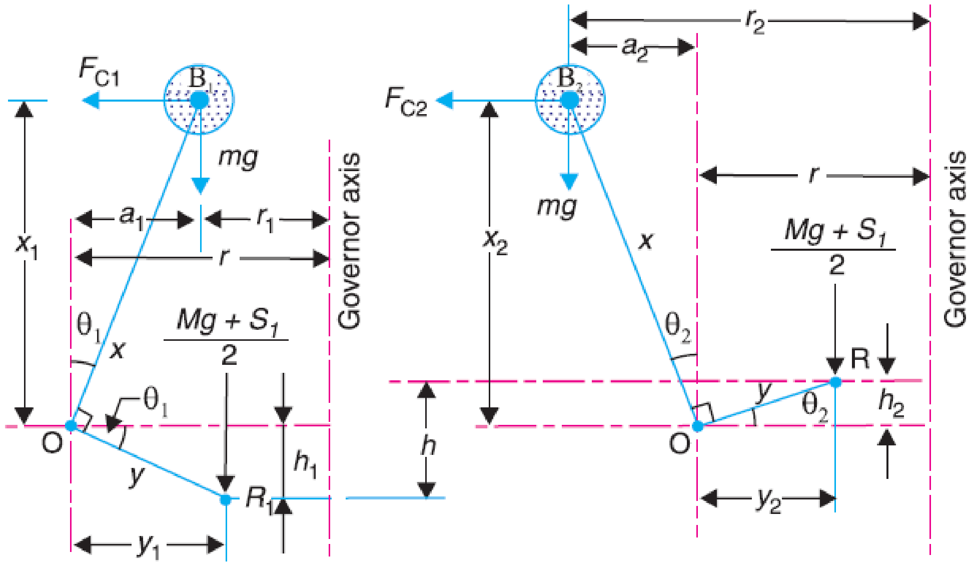
Similarly, for the maximum position *i.e.* when the radius of rotation changes from r to r_2, as shown in Fig (*b*), the compression of the spring or lift of sleeve h_2 is given by

$$\frac{h_2}{y} = \frac{a_2}{x} = \frac{r_2 - r}{x}$$

Adding equations (*i*) and (*ii*),

$$\frac{h_1 + h_2}{y} = \frac{r_2 - r_1}{x} \quad \text{or} \quad \frac{h}{y} = \frac{r_2 - r_1}{x} \qquad \ldots (\because\ h = h_1 + h_2)$$

$$\therefore \qquad h = (r_2 - r_1)\frac{y}{x} \qquad \ldots (iii)$$



(*a*) Minimum position. (*b*) Maximum position.

Now for minimum position, taking moments about point O, we get

$$\frac{M.g + S_1}{2} \times y_1 = F_{C1} \times x_1 - m.g \times a_1$$

$$M.g + S_1 = \frac{2}{y_1}(F_{C1} \times x_1 - m.g \times a_1) \qquad \ldots (iv)$$

Again for maximum position, taking moments about point O, we get

$$\frac{M.g + S_2}{2} \times y_2 = F_{C2} \times x_2 + m.g \times a_2$$

$$M.g + S_2 = \frac{2}{y_2}(F_{C2} \times x_2 + m.g \times a_2) \qquad \ldots (v)$$

Subtracting equation **(*iv*)** from equation **(*v*)**,

$$S_2 - S_1 = \frac{2}{y_2}(F_{C2} \times x_2 + m.g \times a_2) - \frac{2}{y_1}(F_{C1} \times x_1 - m.g \times a_1)$$

We know that

$$S_2 - S_1 = h.s, \qquad \text{and} \qquad h = (r_2 - r_1)\frac{y}{x}$$

$$\therefore \qquad s = \frac{S_2 - S_1}{h} = \left(\frac{S_2 - S_1}{r_2 - r_1}\right)\frac{x}{y}$$

Neglecting the obliquity effect of the arms (*i.e.* $x_1 = x_2 = x$, and $y_1 = y_2 = y$) and the moment due to weight of the balls (*i.e.* $m.g$), we have for minimum position,

$$\frac{M.g + S_1}{2} \times y = F_{C1} \times x \quad \text{or} \quad M.g + S_1 = 2F_{C1} \times \frac{x}{y} \qquad \ldots(vi)$$

Similarly for maximum position,

$$\frac{M.g + S_2}{2} \times y = F_{C2} \times x \quad \text{or} \quad M.g + S_2 = 2F_{C2} \times \frac{x}{y} \qquad \ldots(vii)$$

Subtracting equation ***(vi)*** from equation ***(vii)*,**

$$S_2 - S_1 = 2(F_{C2} - F_{C1})\frac{x}{y} \qquad \ldots(viii)$$

We know that

$$S_2 - S_1 = h.s, \qquad \text{and} \qquad h = (r_2 - r_1)\frac{y}{x}$$

$$s = \frac{S_2 - S_1}{h} = 2\left(\frac{F_{C2} - F_{C1}}{r_2 - r_1}\right)\left(\frac{x}{y}\right)^2 \qquad \ldots(ix)$$

PROBLEMS

Example1. A Hartnell governor having a central sleeve spring and two right-angled bell crank levers moves between 290 r.p.m. and 310 r.p.m. for a sleeve lift of 15 mm. The sleeve arm sand the ball arms are 80 mm and 120 mm respectively. The levers are pivoted at 120 mm from the governor axis and mass of each ball is 2.5 kg. The ball arms are parallel to the governor axis at the lowest equilibrium speed.
Determine:
1. loads on the spring at the lowest and the highest equilibrium speeds, and
2. Stiffness of the spring.

Solution. Given : N_1 = 290 r.p.m. or $\omega_1 = 2\pi \times 290/60 = 30.4$ rad/s ;
N_2 = 310 r.p.m. or $\omega_2 = 2\pi \times 310/60 = 32.5$ rad/s ; h = 15 mm = 0.015 m ;
y = 80 mm = 0.08 m ; x = 120 mm =0.12 m ; r = 120 mm = 0.12 m ; m = 2.5 kg

1. **Loads on the spring at the lowest and highest equilibrium speeds**

Let S= Spring load at lowest equilibrium speed, and

S_2 = Spring load at highest equilibrium speed.

Since the ball arms are parallel to governor axis at the lowest equilibrium speed (*i.e.* atN_1 = 290 r.p.m.), as shown in Fig(*a*), therefore

$r = r_1$ = 120 mm = 0.12 m

We know that centrifugal force at the minimum speed,

$F_{C1} = m (\omega_1)2\ r_1 = 2.5\ (30.4)2\ 0.12 = 277$ N

Now let us find the radius of rotation at the highest equilibrium speed, *i.e.* at N_2 =310 r.p.m. The position of ball arm and sleeve arm at the highest equilibrium speed is shown in Fig (*b*).

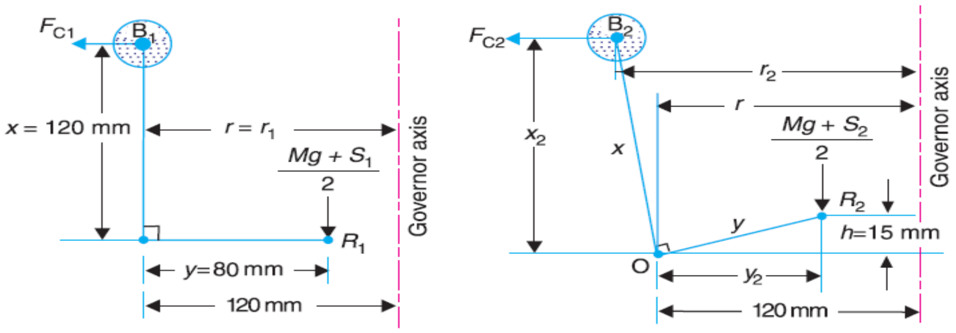
Let r_2 = Radius of rotation at N_2 = 310 r.p.m.

We know that $$h = (r_2 - r_1)\frac{y}{x}$$

$$r_2 = r_1 + h\left(\frac{x}{y}\right) = 0.12 + 0.015\left(\frac{0.12}{0.08}\right) = 0.1425 \text{ m}$$

Centrifugal force at the maximum speed,

$F_{C2} = m (\omega_2)^2\ r_2 = 2.5 \times (32.5)2 \times 0.1425 = 376$ N



(*a*) Lowest position. (*b*) Highest position.

Neglecting the obliquity effect of arms and the moment due to the weight of the balls, we have for lowest position,

$$M.g + S_1 = 2F_{C1} \times \frac{x}{y} = 2 \times 277 \times \frac{0.12}{0.08} = 831 \text{ N}$$

$$S_2 = 831 \text{ N } \textbf{Ans.} \qquad (\because M = 0)$$

and for highest position,

$$M.g + S_2 = 2F_{C2} \times \frac{x}{y} = 2 \times 376 \times \frac{0.12}{0.08} = 1128 \text{ N}$$

$$\therefore \quad S_1 = 1128 \text{ N } \textbf{Ans.} \qquad (\because M = 0)$$

2. **Stiffness of the spring**

We know that stiffness of the spring,

$$s = \frac{S_2 - S_1}{h} = \frac{1128 - 831}{15} = 19.8 \text{ N/mm}$$

Example 2. In a spring loaded Hartnell type governor, the extreme radii of rotation of the balls are 80 mm and 120 mm. The ball arm and the sleeve arm of the bell crank lever are equal in length. The mass of each ball is 2 kg. If the speeds at the two extreme positions are 400 and420 r.p.m.,
find :
1. the initial compression of the central spring, and
2. the spring constant.

Solution. Given : r_1 = 80 mm = 0.08 m ; r_2 = 120 mm = 0.12 m ; $x = y$; m = 2 kg ; N_1 = 400r.p.m. or ω = 2 π× 400/60 = 41.9 rad/s ; N_2 = 420 r.p.m. or ω_2 = 2 π × 420/60 = 44 rad/s

1.Initial compression of the central spring

We know that the centrifugal force at the minimum speed,

$F_{C1} = m\,(\omega_1)^2\, r_1$ = 2 (41.9)2 0.08 = 281 N

and centrifugal force at the maximum speed,

$F_{C2} = m\,(\omega^2)^2\, r_2$ = 2 (44)2 0.12 = 465 N

Let $S1$ = Spring force at the minimum speed, and
$S2$ = Spring force at the maximum speed.

We know that for minimum position,

$$M\,.\,g + S_1 = 2\,F_{C1} \times \frac{x}{y}$$

$$S_1 = 2\,F_{C1} = 2 \times 281 = 562 \text{ N} \qquad \ldots (\because M = 0 \text{ and } x = y)$$

Similarly for maximum position,

$$M\,.\,g + S_2 = 2\,F_{C2} \times \frac{x}{y}$$

$$S_2 = 2\,F_{C2} = 2 \times 465 = 930 \text{ N}$$

We know that lift of the sleeve,

$$h = (r_2 - r_1)\,\frac{y}{x} = r_2 - r_1 = 120 - 80 = 40 \text{ mm} \qquad \ldots (\because x = y)$$

Stiffness of the spring,

$$s = \frac{S_2 - S_1}{h} = \frac{930 - 562}{40} = 9.2 \text{ N/mm}$$

We know that initial compression of the central spring

$$= \frac{S_1}{s} = \frac{562}{9.2} = 61 \text{ mm}$$

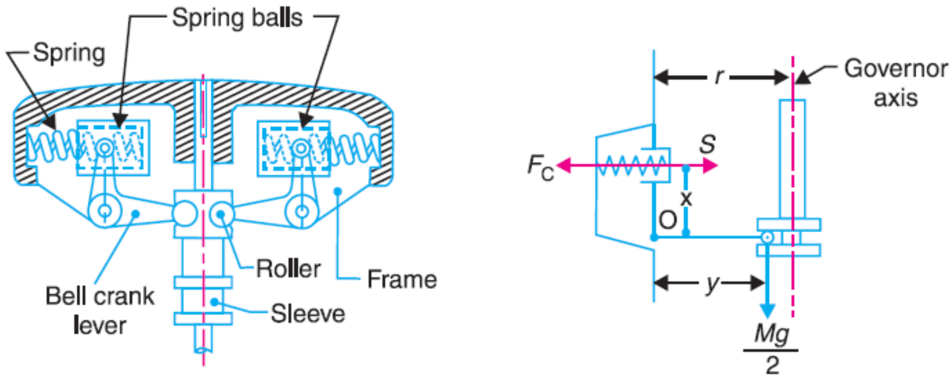
2.Spring constant

We have calculated above that the spring constant or stiffness of the spring, s = 9.2 N/mm

Hartung Governor

A spring controlled governor of the Hartung type is shown in Fig. (*a*). In this type of governor, the vertical arms of the bell crank levers are fitted with spring balls which compress against the frame of the governor when the rollers at the horizontal arm press against the sleeve.

Let S = Spring force,
F_C = Centrifugal force,
M = Mass on the sleeve, and
x and y = Lengths of the vertical and horizontal arm of the bell crank lever respectively.



(*a*) (*b*)

Fig (*a*) and (*b*) show the governor in mid-position. Neglecting the effect of obliquity of the arms, taking moments about the fulcrum O,

$$F_C \times x = S \times x + \frac{M \cdot g}{2} \times y$$

Sensitiveness of Governors

Consider two governors A and B running at the same speed. When this speed increases or decreases by a certain amount, the lift of the sleeve of governor A is greater than the lift of the sleeve of governor B. It is then said that the governor A is more sensitive than the governor B.

In general, the greater the lift of the sleeve corresponding to a given fractional change in speed, the greater is the sensitiveness of the governor. It may also be stated in another way that for a given lift of the sleeve, the sensitiveness of the governor increases as the speed range decreases. This definition of sensitiveness may be quite satisfactory when the governor is considered as an independent mechanism. But when the governor is fitted to an engine, the practical requirement is simply that the change of equilibrium speed from the full load to the no load position of the sleeve should be as small a fraction as possible of the mean equilibrium speed.

The actual displacement of the sleeve is immaterial, provided that it is sufficient to change the energy supplied to the engine by the required amount. For this reason, the sensitiveness is defined as the **ratio of the difference between the maximum and minimum equilibrium speeds to the mean equilibrium speed**

Let N_1 = Minimum equilibrium speed,
N_2 = Maximum equilibrium speed, and
N = Mean equilibrium speed=N_1+N_2/2

∴ Sensitiveness of the governor

$$= \frac{N_2 - N_1}{N} = \frac{2(N_2 - N_1)}{N_1 + N_2}$$

$$= \frac{2(\omega_2 - \omega_1)}{\omega_1 + \omega_2} \quad \text{. . . (In terms of angular speeds)}$$

Stability of Governors

A governor is said to be ***stable*** when for every speed within the working range there is a definite configuration *i.e.* there is only one radius of rotation of the governor balls at which the governor is in equilibrium. For a stable governor, if the equilibrium speed increases, the radius of governor balls must also increase.

Isochronous Governors

A governor is said to be ***isochronous*** when the equilibrium speed is constant (*i.e.* range of speed is zero) for all radii of rotation of the balls within the working range, neglecting friction. The isochronisms is the stage of infinite sensitivity.
Let us consider the case of a Porter governor running at speeds $N1$ and $N2$ r.p.m.

$$(N_1)^2 = \frac{m + \frac{M}{2}(1+q)}{m} \times \frac{895}{h_1} \quad \text{. . . }(i)$$

$$(N_2)^2 = \frac{m + \frac{M}{2}(1+q)}{m} \times \frac{895}{h_2} \quad \text{. . . }(ii)$$

For isochronisms, range of speed should be zero *i.e.* $N_2 - N_1 = 0$ or $N_2 = N_1$. Therefore from equations **(*i*)** and **(*ii*),** $h_1 = h_2$, which is impossible in case of a Porter governor. Hence a ***Porter governor cannot be isochronous.***

Now consider the case of a Hartnell governor running at speeds N_1 and N_2 r.p.m.

$$M.g + S_1 = 2\,F_{C1} \times \frac{x}{y} = 2 \times m \left(\frac{2\pi N_1}{60}\right)^2 r_1 \times \frac{x}{y} \quad \text{. . . }(iii)$$

$$M.g + S_2 = 2\,F_{C2} \times \frac{x}{y} = 2 \times m \left(\frac{2\pi N_2}{60}\right)^2 r_2 \times \frac{x}{y} \quad \text{. . . }(iv)$$

For isochronisms, $N_2 = N_1$. Therefore from equations **(*iii*)** and **(*iv*),**

$$\frac{M.g + S_1}{M.g + S_2} = \frac{r_1}{r_2}$$

Hunting

A governor is said to be ***hunt*** if the speed of the engine fluctuates continuously above and below the mean speed. This is caused by a too sensitive governor which changes the fuel supply by a large amount when a small change in the speed of rotation takes place. For example, when the load on the engine increases, the engine speed decreases and, if the governor is very sensitive ,the governor sleeve immediately falls to its lowest position. This will result in the opening of the control valve wide which will supply the fuel to the engine in excess of its requirement so that the engine speed rapidly increases again and the governor sleeve rises to its highest position. Due to this movement of the sleeve, the control valve will cut off the fuel supply to the engine and thus the engine speed begins to fall once again. This cycle is repeated indefinitely. Such a governor may admit either the maximum or the minimum amount of fuel. The effect of this will be to cause wide fluctuations in the engine speed or in other words, the engine will hunt